

Contents

Preface — XIII

Introduction — 1

1 On the way to the reals — 7

1.1 Irrationality — 7

1.2 Incommensurability — 9

1.3 Calculating with $\sqrt{2}$? — 13

1.4 Procedure of approximating, nesting of intervals and completeness — 14

1.5 On the construction of the reals — 19

1.6 On the handling of the infinite — 21

1.7 Infinite non-periodic decimal fractions — 23

2 On the history of the philosophy of mathematics — 27

2.1 Pythagoras and Pythagoreans — 28

2.2 Plato — 31

2.3 Aristotle — 34

2.4 Euclid — 38

2.5 Proclus Diadochus — 40

2.6 Nicholas of Cusa — 42

2.7 Descartes — 45

2.8 Pascal — 48

2.9 Leibniz — 50

2.10 Kant — 53

2.11 Mill and the empirical conceptions — 58

2.12 Bolzano — 62

2.13 Gauß — 65

2.14 Cantor — 66

2.15 Dedekind — 70

2.16 Poincaré — 75

2.17 Peirce's pragmatism and the world of symbols — 78

2.18 Husserl's phenomenology — 82

2.19 Logicism — 89

2.20 Intuitionism — 98

2.21 Constructivism — 107

2.22 Formalism — 109

2.23 Philosophy of mathematics between 1931 and the end of the 1950s — 117

2.24	The evolutionary point of view – a new basic position in philosophy — 123
2.24.1	Characterization — 123
2.24.2	On studies of evolution of number concept — 127
2.24.3	Concluding remarks — 136
2.25	Philosophy of mathematics after 1960 — 137
2.25.1	Quasi-empirical conceptions — 138
2.25.2	Realism and antirealism — 146
3	On fundamental questions of the philosophy of mathematics — 149
3.1	On the concept of number — 149
3.1.1	Survey of some views — 150
3.1.2	Résumé — 154
3.2	Infinites — 159
3.2.1	On problems with the infinite — 160
3.2.2	Conception of Aristotle — 163
3.2.3	Idealistic approach — 163
3.2.4	Empiricist point of view — 164
3.2.5	Infinity by Kant — 165
3.2.6	Intuitionistic infinity — 167
3.2.7	Logicist hypothesis of the infinite — 167
3.2.8	Infinity and the new philosophy of mathematics — 168
3.2.9	Formalistic approach and nowadays tendencies — 169
3.3	The continuum and the infinitely small — 170
3.3.1	General problem — 171
3.3.2	On the history of the continuum — 173
3.3.3	What is a point? — 184
3.3.4	On the history of the continuum – continuation — 190
3.3.5	Survey of conceptions of the continuum — 194
3.3.6	Notes on the arithmetization of the continuum — 196
3.3.7	The end of infinitesimals and the rediscovery of them — 198
3.3.8	Nonstandard numbers and the continuum — 205
3.3.9	Consequences for the conception of the continuum — 207
3.3.10	Medium of free evolution — 210
3.3.11	The disappearance of magnitudes — 212
3.3.12	Final remarks — 217
3.4	On the problem of applications of mathematics — 222
3.4.1	Aspects of the problem — 223
3.4.2	The problem of applicability in the historical setting — 227
3.4.3	Classical positions — 233
3.4.4	New conceptions — 236
3.4.5	Retrospect — 236

3.5	Conclusion — 241
3.5.1	From natural to rational numbers — 242
3.5.2	Incommensurability and irrationality — 243
3.5.3	Adjunction — 245
3.5.4	On the linear continuum — 246
3.5.5	Infinitely small quantities — 247
3.5.6	Construction, infinity, infinite non-periodic decimal fractions — 248
3.5.7	Concluding remarks — 249
4	Sets and set theories — 253
4.1	Paradoxes of the infinite — 254
4.2	On the concept of a set — 256
4.2.1	Collecting together versus putting together — 256
4.2.2	Sets and the problem of universals — 257
4.3	Two set theories — 260
4.3.1	Set theory according to Zermelo and Fraenkel — 261
4.3.2	Von Neumann, Bernays and Gödel set theory — 269
4.3.3	Remarks — 274
4.3.4	On modifications — 275
4.4	The Axiom of Choice and the Continuum Hypothesis — 276
4.4.1	Search for new axioms — 281
4.4.2	Further remarks and questions — 286
4.5	Final remarks — 287
5	Axiomatic approach and logic — 293
5.1	Some elements of mathematical logic — 294
5.1.1	Syntax — 294
5.1.2	Semantics — 296
5.1.3	Calculus — 299
5.2	Historical remarks — 301
5.2.1	From the history of logic — 302
5.2.2	On the history of the axiomatic approach — 310
5.3	Logical axioms and theories — 315
5.3.1	Peano arithmetic — 317
5.3.2	On the axioms for real numbers — 320
5.4	On the arithmetic of natural numbers — 323
5.4.1	Syntactical aspect — 324
5.4.2	Semantical aspect — 326
5.5	Truth and provability — 330
5.5.1	Formal truth — 331
5.5.2	Completeness and truth — 332
5.5.3	Syntactic reduction of truth — 334

5.5.4	Truth is unequal to provability —	337
5.5.5	Search for the way out —	339
5.5.6	Concluding remarks —	341
5.6	Final conclusions —	341
5.6.1	Logic as the background of mathematics —	342
5.6.2	Consequences for the shape of mathematics —	343
6	Thinking and calculating infinitesimally – First nonstandard steps —	347
6.1	Preliminary remark —	347
6.2	The question about $0.999 \dots$ —	348
6.2.1	Empirical approach —	348
6.2.2	The problem —	349
6.2.3	Answer —	356
6.2.4	Final remarks —	361
6.3	A bit of infinitesimal calculus —	362
6.3.1	Infinitesimal calculations —	363
6.3.2	Continuity, differential quotient, derivative —	365
6.4	On the construction of hyperreal numbers —	374
6.4.1	Hypersnatural numbers —	375
6.4.2	Hyperreal numbers —	376
6.4.3	How will ${}^*\mathbb{R}$ become a model of $\mathbb{R}$? —	378
6.4.4	On the justification of the naive infinitesimal calculus —	379
6.5	On the status of nonstandard numbers —	380
7	Retrospection —	387
7.1	Setting of real numbers —	388
7.2	Axiomatic method —	389
7.3	The concept of number —	390
7.4	Infinity, Axiom of Choice and Continuum Hypothesis —	392
7.5	Infinitely small and the continuum —	393
7.6	Applicability —	395
7.7	Theoretical limits —	395
7.8	Usage of computers —	396
7.9	What is the philosophy of mathematics and what does it provide? —	397
7.10	Evidence and transcendence —	399

Biographies — 403

Bibliography — 421

Index of names — 437

Index of symbols — 443

Index of subjects — 445