

# Contents

## Preface — VII

## Part I: Basic methods

### Introduction — 3

|          |                                                                                                             |
|----------|-------------------------------------------------------------------------------------------------------------|
| <b>0</b> | <b>Infinite lower triangular matrices and formal power series — 5</b>                                       |
| 0.1      | Definitions and first properties — 5                                                                        |
| 0.1.1    | Hessenberg determinants — 7                                                                                 |
| 0.2      | Triangular Toeplitz matrices — 11                                                                           |
| 0.3      | A class of tridiagonal matrices and the related determinant computation — 13                                |
| 0.4      | Formal power series: algebraic structure — 17                                                               |
| 0.5      | The $\delta$ -series and compositional inverse — 19                                                         |
| 0.6      | Formal power series with only even powers and related Toeplitz matrices — 25                                |
| 0.7      | Appell-type matrices — 26                                                                                   |
| 0.8      | Sheffer-type matrices — 34                                                                                  |
| 0.9      | Lidstone-type matrices — 40                                                                                 |
| 0.9.1    | Odd Lidstone-type matrices — 40                                                                             |
| 0.9.2    | Even Lidstone-type matrices — 44                                                                            |
| 0.9.3    | Relationship between Appell-type and Lidstone-type matrices — 47                                            |
| 0.10     | Production or Stieltjes matrix — 47                                                                         |
| 0.11     | Hankel matrices — 49                                                                                        |
| <b>1</b> | <b>Polynomial sequences: algebraic structure, recurrence and determinant form, operational methods — 56</b> |
| 1.1      | The linear space $\widehat{\mathcal{P}}$ — 56                                                               |
| 1.2      | Recurrence relations — 59                                                                                   |
| 1.3      | Determinant forms — 62                                                                                      |
| 1.4      | Basis, connection constants and condition number — 66                                                       |
| 1.5      | Illustrative examples — 68                                                                                  |
| 1.5.1    | Fibonacci polynomial sequences — 71                                                                         |
| 1.5.2    | Chebyshev polynomial sequences of the first kind — 78                                                       |
| 1.6      | Comparison between condition numbers — 86                                                                   |
| 1.7      | Operational matrices — 88                                                                                   |
| 1.8      | Derivation matrix: calculation — 88                                                                         |
| 1.9      | Integration matrix: calculation — 90                                                                        |
| 1.10     | Illustrative examples — 92                                                                                  |

**2 Symmetric polynomial sequences — 97**

- 2.1 Definitions — 97
- 2.2 Matrix form — 98
- 2.3 The algebraic structure — 99
- 2.4 Recurrence relations — 100
- 2.5 Determinant forms — 102
- 2.6 Illustrative examples — 105
  - 2.6.1 Odd and even Fibonacci polynomial sequences — 105
  - 2.6.2 Odd and even Chebyshev polynomial sequences of the first kind — 109
  - 2.6.3 Odd and even central factorial polynomial sequences — 113

**3 Generating functions — 117**

- 3.1 Definitions — 117
- 3.2 Series manipulation — 118
- 3.3 The factorial function and the generalized hypergeometric functions — 119
- 3.4 Obtaining generating functions by representation in monomial power — 121

**4 Differential operator and Sheffer classification — 129**

- 4.1 The polynomial set associated with a polynomial sequence — 129
- 4.2 Differential operator associated with polynomial sequence: Sheffer's classification — 131
- 4.3 Illustrative examples — 133
  - 4.3.1 Monic Laguerre polynomials — 133
  - 4.3.2 Laguerre polynomials — 134
  - 4.3.3 Hermite polynomials — 135
  - 4.3.4 Appell polynomial sequences — 135
- 4.4 Appell-type 1 polynomial sequences — 136
  - 4.4.1 Matrix form — 138
  - 4.4.2 Recurrence relations and determinant forms — 139
  - 4.4.3 Particular cases — 140

**5 The monomiality principle — 143**

- 5.1 Introduction — 143
- 5.2 The derivative and multiplicative operators — 144
- 5.3 Illustrative examples — 146

## Part II: Special classes of polynomial sequences

### Introduction — 155

#### 6 Sheffer polynomial sequences — 157

- 6.1 Introduction — 157
- 6.2 Definitions and first characterizations — 158
- 6.3 Sheffer classification — 162
- 6.4 Algebraic structure of the set  $\mathcal{G}$  — 164
- 6.5 Matrix, recurrence and determinant forms — 166
- 6.6 Generating function — 168
- 6.7 General properties — 170
- 6.8 Relationship with monomiality principle — 171
- 6.9 Differential equations — 172
- 6.10 Relationship with linear functional — 173
- 6.11 Some special Sheffer polynomial sequences — 174
  - 6.11.1 Generalized Laguerre polynomials — 175
  - 6.11.2 Hermite polynomials — 180
  - 6.11.3 Lower factorial and exponential polynomials — 183
  - 6.11.4 Central factorial and Carlitz–Riordan polynomials — 189
  - 6.11.5 Generalized falling factorial and Appell-based falling factorial sequences — 194
- 6.12 Orthogonal Sheffer polynomial sequences — 200
  - 6.12.1 Introduction — 200
  - 6.12.2 A matrix calculus-based approach to determine the previous Sheffer results — 202
  - 6.12.3 Orthogonal  $A$ -type 0 polynomials — 203

#### 7 Orthogonal polynomial sequences — 207

- 7.1 Introduction — 207
- 7.2 Definitions and some properties — 209
- 7.3 Monic and orthonormal polynomials — 217
- 7.4 The symmetric case — 219
- 7.5 Illustrative examples — 220
- 7.6 Algorithms for numerically generating orthogonal polynomials — 227
  - 7.6.1 Chebyshev algorithm — 227
  - 7.6.2 New algorithms — 229
- 7.7 Numerical examples — 231

#### 8 Lidstone and central factorial-type polynomial sequences — 238

- 8.1 Introduction — 238
- 8.2 Odd Lidstone-type polynomial sequences — 239

|           |                                                                               |
|-----------|-------------------------------------------------------------------------------|
| 8.2.1     | Matrix form — <b>240</b>                                                      |
| 8.2.2     | First recurrence relations and related determinant forms — <b>242</b>         |
| 8.2.3     | Second recurrence relation and related determinant form — <b>243</b>          |
| 8.2.4     | Differential equations for odd Lidstone-type polynomials — <b>246</b>         |
| 8.2.5     | Operational matrices — <b>247</b>                                             |
| 8.2.6     | Generating function — <b>250</b>                                              |
| 8.2.7     | Relationship with Appell polynomial sequences — <b>252</b>                    |
| 8.2.8     | Examples — <b>253</b>                                                         |
| 8.3       | Even Lidstone-type polynomial sequences — <b>259</b>                          |
| 8.3.1     | Matrix form — <b>260</b>                                                      |
| 8.3.2     | First recurrence relation and related determinant form — <b>262</b>           |
| 8.3.3     | Second recurrence relation and related determinant form — <b>263</b>          |
| 8.3.4     | Differential equations for even Lidstone-type polynomials — <b>264</b>        |
| 8.3.5     | Operational matrices — <b>264</b>                                             |
| 8.3.6     | Generating function — <b>265</b>                                              |
| 8.3.7     | Relationship with Appell polynomial sequences — <b>266</b>                    |
| 8.3.8     | Examples — <b>266</b>                                                         |
| 8.4       | General odd and even central factorial-type polynomial sequences — <b>271</b> |
| 8.5       | General odd central factorial-type polynomial sequences — <b>272</b>          |
| 8.5.1     | Matrix form — <b>273</b>                                                      |
| 8.5.2     | Conjugate polynomial sequences — <b>274</b>                                   |
| 8.5.3     | Recurrence relations and related determinant forms — <b>275</b>               |
| 8.5.4     | Exponential generating function — <b>277</b>                                  |
| 8.5.5     | Connection to the basis monomials $x^{2j+1}$ — <b>278</b>                     |
| 8.5.6     | Examples — <b>280</b>                                                         |
| 8.6       | General even central factorial polynomial sequences — <b>283</b>              |
| 8.6.1     | Matrix form — <b>284</b>                                                      |
| 8.6.2     | Conjugate even polynomials — <b>285</b>                                       |
| 8.6.3     | Recurrence relation and related determinant form — <b>286</b>                 |
| 8.6.4     | Generating function — <b>287</b>                                              |
| 8.6.5     | Connection to the basis monomials $x^{2j}$ — <b>288</b>                       |
| 8.6.6     | Examples — <b>289</b>                                                         |
| <b>9</b>  | <b>Bernstein basis — 293</b>                                                  |
| 9.1       | Introduction — <b>293</b>                                                     |
| 9.2       | Bernstein basis — <b>294</b>                                                  |
| 9.3       | Numerical stability — <b>304</b>                                              |
| 9.4       | Basis transformations — <b>306</b>                                            |
| 9.5       | Generating function — <b>308</b>                                              |
| <b>10</b> | <b>Bivariate special polynomials: hints — 310</b>                             |
| 10.1      | Introduction — <b>310</b>                                                     |

|        |                                                             |     |
|--------|-------------------------------------------------------------|-----|
| 10.2   | General bivariate Appell polynomials —                      | 311 |
| 10.2.1 | Definitions and first properties —                          | 311 |
| 10.2.2 | Matrix form —                                               | 315 |
| 10.2.3 | Recurrence relations —                                      | 318 |
| 10.2.4 | Determinant forms —                                         | 319 |
| 10.2.5 | Differential operators and equations —                      | 322 |
| 10.2.6 | Relationship with monomiality principles —                  | 322 |
| 10.2.7 | General properties —                                        | 323 |
| 10.2.8 | Relations with linear functional and linear interpolation — | 326 |
| 10.2.9 | Some bivariate Appell sequences —                           | 327 |
| 10.3   | General bivariate Lidstone-type polynomials —               | 340 |
| 10.3.1 | Introduction —                                              | 340 |
| 10.3.2 | General bivariate odd Lidstone-type polynomial sequences —  | 340 |
| 10.3.3 | Matrix form —                                               | 344 |
| 10.3.4 | Recurrence relations and related determinant forms —        | 345 |
| 10.3.5 | Differential equation —                                     | 346 |
| 10.3.6 | General properties —                                        | 346 |
| 10.3.7 | Some bivariate odd Lidstone-type sequences —                | 349 |
| 10.4   | Bivariate Bernstein basis on the simplex —                  | 351 |

## Part III: Computational applications

### Introduction — 359

|           |                                                                                                   |            |
|-----------|---------------------------------------------------------------------------------------------------|------------|
| <b>11</b> | <b>Approximation theory by operators —</b>                                                        | <b>361</b> |
| 11.1      | Bernstein and Bernstein-type operators —                                                          | 361        |
| 11.1.1    | Bernstein's theorem —                                                                             | 362        |
| 11.1.2    | Asymptotic expansion —                                                                            | 366        |
| 11.1.3    | Some algorithms of polynomial extrapolation —                                                     | 367        |
| 11.1.4    | Butzer and May polynomials —                                                                      | 369        |
| 11.1.5    | Two interesting applications —                                                                    | 370        |
| 11.1.6    | Numerical examples —                                                                              | 371        |
| 11.1.7    | Recent research —                                                                                 | 373        |
| 11.1.8    | Binomial-type or Bernstein–Sheffer operators —                                                    | 374        |
| 11.2      | Approximation over a semi-infinite interval: Szász and Szász-type operators —                     | 376        |
| 11.2.1    | Asymptotic expansion and extrapolation for operators $F_n$ —                                      | 378        |
| 11.2.2    | Special cases: Examples of operators. Operators of Jakimovski and Leviatan-type and Ismail-type — | 379        |
| 11.2.3    | New operators of Jakimovski and Leviatan type —                                                   | 381        |
| 11.2.4    | New operators of Ismail type —                                                                    | 382        |

|           |                                                                                                 |            |
|-----------|-------------------------------------------------------------------------------------------------|------------|
| 11.2.5    | Comparison of rate of convergence —                                                             | <b>384</b> |
| 11.3      | Numerical examples —                                                                            | <b>385</b> |
| <b>12</b> | <b>Interpolation —</b>                                                                          | <b>390</b> |
| 12.1      | Preliminaries —                                                                                 | <b>390</b> |
| 12.2      | Umbral basis for $(L, Q)$ —                                                                     | <b>391</b> |
| 12.3      | Umbral interpolation —                                                                          | <b>392</b> |
| 12.4      | Examples —                                                                                      | <b>395</b> |
| 12.4.1    | Appell interpolation —                                                                          | <b>395</b> |
| 12.4.2    | $\Delta_h$ -Appell interpolation —                                                              | <b>398</b> |
| 12.4.3    | $\nabla^h, \delta_h$ -Appell interpolation —                                                    | <b>401</b> |
| 12.4.4    | Abel-Sheffer interpolation [116] —                                                              | <b>401</b> |
| 12.4.5    | $\delta_h$ -Sheffer interpolation —                                                             | <b>403</b> |
| 12.5      | Summary of umbral interpolation formulas —                                                      | <b>404</b> |
| 12.6      | Lidstone-type polynomial interpolation: preliminaries —                                         | <b>404</b> |
| 12.7      | The Lidstone-type and the generalized Lidstone interpolation problem —                          | <b>410</b> |
| 12.8      | Explicit odd and even Lidstone-type interpolating polynomials —                                 | <b>412</b> |
| 12.9      | Examples —                                                                                      | <b>415</b> |
| 12.9.1    | Lidstone-Bernoulli-type interpolating polynomials —                                             | <b>416</b> |
| 12.9.2    | Lidstone-Euler-type interpolating polynomials —                                                 | <b>421</b> |
| 12.10     | Summary of Lidstone interpolation formulas —                                                    | <b>425</b> |
| 12.11     | Numerical examples —                                                                            | <b>425</b> |
| 12.12     | Odd and even central difference-type interpolation —                                            | <b>429</b> |
| <b>13</b> | <b>Boundary value problems and polynomial sequences —</b>                                       | <b>430</b> |
| 13.1      | Introduction —                                                                                  | <b>430</b> |
| 13.2      | Spectral methods —                                                                              | <b>431</b> |
| 13.3      | A spectral method for a class of linear boundary value problems —                               | <b>431</b> |
| 13.3.1    | Bernoulli polynomial sequences for the numerical solution of general second-order linear BVPs — | <b>431</b> |
| 13.3.2    | The method —                                                                                    | <b>433</b> |
| 13.3.3    | Numerical computation of the solution —                                                         | <b>438</b> |
| 13.3.4    | Numerical examples —                                                                            | <b>442</b> |
| 13.4      | General nonlinear high-order BVPs: Birkhoff-Lagrange collocation methods —                      | <b>447</b> |
| 13.4.1    | A priori estimation of error —                                                                  | <b>450</b> |
| 13.4.2    | Algorithms and implementation —                                                                 | <b>451</b> |
| 13.4.3    | Special cases —                                                                                 | <b>454</b> |
| 13.4.4    | Numerical examples —                                                                            | <b>458</b> |

**14 Appell and Lidstone-type quadrature formulas — 463**

14.1 Introduction — **463**

14.2 Appell quadrature rules — **463**

14.2.1 Appell composite quadrature formulas — **466**

14.2.2 Examples — **469**

14.3 Lidstone-type quadrature rules — **475**

14.3.1 Examples — **478**

**Postface — 485**

**Bibliography — 487**

**Index — 503**