

# Contents

## Introduction — V

## List of notations and abbreviations — XIII

Operators — XIII

Spaces of functions — XIII

Special functions — XIV

Stochastic processes and random variables — XV

Other symbols — XVI

Abbreviations — XVII

## 1 Fractional integrals and derivatives — 1

- 1.1 General properties of fractional integrals — 1
- 1.2 Hölder property of fractional integrals — 10
- 1.3 Fractional integrals of power-integrable functions — 15
- 1.4 Restrictions of fractional integrals — 18
- 1.5 Continuity of fractional integrals in the index of integration — 20
- 1.6 Fractional integrals of complex order — 23
- 1.7 Riemann–Liouville fractional derivatives — 24
- 1.8 Dzhrbashyan–Caputo fractional derivatives — 28
- 1.9 Marchaud fractional derivatives — 32
- 1.10 Fractional derivatives of higher order — 42
- 1.11 The one-dimensional fractional Laplacian — 64
- 1.12 Grunwald–Letnikov fractional derivatives — 85
- 1.13 Fractional derivatives of complex order — 88
- 1.14 Fractional integrals and derivatives with respect to a function — 88
- 1.15 Further properties of fractional derivatives — 92
- 1.16 The Hilbert transform — 101
- 1.17 Exercises — 108

## 2 Integral and differential equations involving fractional operators — 120

- 2.1 Abel integral equation — 120
- 2.2 The Mittag-Leffler and Prabhakar functions — 121
- 2.3 Fractional differential equations with Riemann–Liouville fractional derivatives — 130
- 2.4 Fractional differential equations with Dzhrbashyan–Caputo fractional derivatives — 143
- 2.5 The generalized Grönwall inequality — 151
- 2.6 Continuous dependence on the initial data — 154
- 2.7 Linear fractional differential equations with Riemann–Liouville fractional derivatives — 157

2.8	Linear fractional differential equations with Dzhrbashyan–Caputo fractional derivatives — <b>173</b>
2.9	Exercises — <b>181</b>
<b>3</b>	<b>Fractional Brownian motion and its environment — 191</b>
3.1	Fractional Brownian motion: definition and some properties — <b>191</b>
3.2	Wiener integration with respect to the fractional Brownian motion — <b>194</b>
3.3	Wiener integrals of functions of bounded variation — <b>202</b>
3.4	Representations of the fractional Brownian motion — <b>206</b>
3.5	Fractional Ornstein–Uhlenbeck process — <b>214</b>
3.6	Fractional Ornstein–Uhlenbeck process with stochastic forcing — <b>223</b>
3.7	Reflected fractional Brownian motion — <b>229</b>
3.8	Reflected fractional Ornstein–Uhlenbeck process — <b>235</b>
3.9	Exercises — <b>252</b>
<b>4</b>	<b>Stochastic processes and fractional differential equations — 259</b>
4.1	Stable distributions and stable processes — <b>259</b>
4.2	Stable subordinators and Marchaud fractional derivatives — <b>264</b>
4.3	Symmetric $\alpha$ -stable Lévy motion and fractional Laplacian — <b>270</b>
4.4	Inverse stable subordinators — <b>273</b>
4.5	Time-fractional abstract Cauchy problems — <b>279</b>
4.6	Weighted inverse subordination operators — <b>284</b>
4.7	The delayed Brownian motion and the time-fractional heat equation — <b>292</b>
4.8	The delayed fractional Brownian motion — <b>301</b>
4.9	The delayed fractional Ornstein–Uhlenbeck process — <b>313</b>
4.10	Delayed continuous-time Markov chains — <b>316</b>
4.11	Fractional integral equations with a stochastic driver — <b>326</b>
4.12	Exercises — <b>333</b>
<b>5</b>	<b>Numerical methods and simulation — 338</b>
5.1	Calculation of the Mittag-Leffler function — <b>338</b>
5.2	Approximation of fractional integrals. Product-integration methods — <b>345</b>
5.3	Product-integration methods for fractional differential equations — <b>350</b>
5.4	Fractional linear multistep methods — <b>358</b>
5.5	Simulation of the fractional Brownian motion — <b>363</b>
5.6	Euler–Maruyama schemes for stochastic differential equations driven by an additive fractional Brownian noise — <b>368</b>
5.7	Chambers–Mallow–Stuck algorithm — <b>376</b>
5.8	Simulation of the delayed processes — <b>378</b>
5.9	Listings — <b>389</b>

**A Basics in complex analysis and integral transforms — 399**

A.1 Basics in complex analysis — 399

A.2 Fourier transform — 403

A.3 Laplace transform — 408

**B Special functions — 411**

B.1 Euler Gamma function — 411

B.2 Generalized binomial coefficients — 413

B.3 Euler Beta function — 414

B.4 Hypergeometric series — 415

**C Stochastic processes — 417**

C.1 The Kolmogorov–Chentsov theorem — 417

C.2 Gaussian processes — 417

C.3 Brownian motion and integrals with respect to it — 419

C.4 Infinite divisibility and Lévy processes — 420

C.5 Feller semigroups and generators — 424

**Bibliography — 429**

**Index — 439**