

Table of Contents

1. Basic mechanics	1
1.1 On tensors	1
1.2 Stress	3
1.3 Strain	5
1.4 Principal invariants and eigenvalues of stress and strain	6
1.5 Mohr's stress circles	7
1.6 Equilibrium	8
1.7 Local formulation of static problems	9
1.8 Continuity equations	11
1.9 Compatibility equations	12
1.10 Strength criteria	12
1.11 Linear elasticity	16
1.12 Strain energy	21
1.13 Navier equations	23
1.14 Beltrami-Mitchell compatibility equations	24
1.15 Saint-Venant's principle, Uniqueness, Superposition, Special theories	25
1.16 Solved problem: composite cylinder under axial load	26
2. Variational formulations, work and energy theorems	29
2.1 Local formulation of static problems	29
2.2 Virtual work theorem (VWT)	30
2.3 Displacement-based variational formulation	33
2.4 Potential energy theorem	35
2.5 Stress-based variational formulation	37
2.6 Complementary energy theorem	38
2.7 Energy bounds	39
2.8 Stored energy	40
2.9 Maxwell-Betti reciprocity theorem	41
2.10 Castigliano's theorem	42
2.11 Introduction to numerical methods	45
2.11.1 Method of Ritz	45

2.11.2	Finite element method (FEM)	45
2.12	Solved problem: volume change of an axially compressed body	47
3.	Theory of beams (strength of materials)	49
3.1	Definitions and geometric properties	49
3.2	Pure bending of a straight beam: 3D elasticity solution	51
3.3	Basic assumptions of beam theory	55
3.3.1	External loads	56
3.3.2	Internal loads (stress resultants)	57
3.3.3	Equilibrium equations	58
3.3.4	Navier-Bernoulli assumption	59
3.3.5	Constitutive equations	59
3.4	Displacement boundary and continuity conditions	60
3.5	Computation of internal loads (stress resultants)	62
3.6	Computation of normal and shear stresses	64
3.7	Statically determinate or indeterminate problems	66
3.8	Strain energy	67
3.9	Work and energy theorems	68
3.10	Influence lines	69
3.11	Solved problems	70
3.11.1	Shear reduced area	70
3.11.2	Statically determinate problems	71
3.11.3	Statically indeterminate problems	77
3.11.4	Maxwell-Betti reciprocity theorem	80
3.11.5	Castigliano's theorem	85
3.11.6	Virtual work theorem (VWT)	90
4.	Torsion of beams	95
4.1	Formulation with a w^T arping function	95
4.2	Formulation with a conjugate function	98
4.3	Formulation with Prandtl's stress function	98
4.4	Contour lines of the stress function	100
4.5	Maximum tangential stress	101
4.6	Membrane analogy	102
4.7	Multi-connected sections- Particular case	104
4.8	Multi-connected sections- General case	104
4.9	Thin tubes with variable thickness	107
4.10	Potential energy	108
4.11	Cylindrical coordinates	109
4.12	Solved problems	110
4.12.1	Elliptic section	110
4.12.2	Triangular section	113
4.12.3	Notched circular section	114
4.12.4	Thin rectangular section	116
4.12.5	Ritz's method- Square section	117

4.12.6	Hollow elliptic section- Special method	118
4.12.7	Hollow elliptic section- General method	119
4.12.8	Thin circular tube	121
4.12.9	Thin-walled section with multiple voids.	121
5.	Theory of thin plates	123
5.1	Definitions and notation	123
5.2	Internal loads (stress resultants).	123
5.3	Equilibrium equations.	126
5.4	Displacements.	127
5.5	Strains.	129
5.6	Constitutive equations.	129
5.7	Summary: two un-coupled problems.	130
5.8	Fundamental P.D.E. for bending problem.	131
5.9	Boundary conditions.	133
5.10	Contradictions in Kirchhoff-Love theory.	135
5.11	Plates with two simply supported opposite edges - Lévy's method	136
5.12	Potential energy.	139
5.13	Influence function.	140
5.14	Solved problems.	140
5.14.1	Uniformly loaded rectangular plate with two simply supported opposite edges and two built-in edges.	140
5.14.2	Uniformly loaded rectangular plate with two simply supported opposite edges and two free edges.	142
6.	Bending of thin plates in polar coordinates	143
6.1	Change of coordinates.	143
6.2	Axisymmetric problems.	146
6.3	Potential energy.	148
6.4	Solved problems.	149
6.4.1	Uniformly loaded plate.	149
6.4.2	Uniform load along a concentric circle.	150
6.4.3	Uniform pressure on a concentric disk.	153
6.4.4	Plate simply supported on a number of points.	155
6.4.5	Ritz's method.	160
7.	Two-dimensional problems in Cartesian coordinates.	163
7.1	Plane strain.	163
7.2	Plane stress.	164
7.3	Summary: plane strain versus plane stress.	165
7.4	Airy stress function.	167
7.5	Polynomial solutions.	167
7.6	Solution by Fourier series.	169
7.7	Generalized plane stress.	170

7.8	Solved problems.	173
7.8.1	Concentrated load at the end of a cantilever beam	173
7.8.2	Uniform loads on the upper and lower surfaces of a simply supported beam.	177
7.8.3	Uniform load on a cantilever beam.	182
7.8.4	Uniform load on a beam with two clamped ends.	182
7.8.5	Compression of a beam in the height-direction.	182
7.8.6	Body forces- Beam under its own weight.	187
8.	Two-dimensional problems in polar coordinates.	193
8.1	Change of coordinates.	193
8.2	Summary: plane strain versus plane stress.	195
8.3	Airy stress function.	196
8.4	Axisymmetric plane problems.	198
8.5	Periodic Airy stress functions.	200
8.6	Generalized plane strain.	201
8.7	Solved problems.	201
8.7.1	Hollow circular cylinder under inner and outer pressures	201
8.7.2	Composite hollow cylinder under inner and outer pressures.	205
8.7.3	Coil winding.	206
8.7.4	Bending of a curved beam.	209
8.7.5	Traction of a circular arch.	214
8.7.6	Rotating disk of uniform thickness.	215
8.7.7	Rotating disk of variable thickness.	218
8.7.8	Stress concentration in a plate with a small circular hole	222
8.7.9	Force on the straight edge of a semi-infinite plate. . . .	224
8.7.10	Pressure on the straight edge of a semi-infinite plate. . .	228
8.7.11	Compression of a disk along a diameter.	230
8.7.12	Compression of a disk over two opposing arcs.	231
9.	Thermo-elasticity.	233
9.1	Constitutive equations.	233
9.2	Heat equation.	234
9.3	Thermo-mechanical problem.	235
9.3.1	Thermal problem.	236
9.3.2	Mechanical problem.	237
9.4	Thermal stresses: some remarks.	237
9.5	Solved problems.	239
9.5.1	Axisymmetric thermal stresses in a hollow cylinder. . .	239
9.5.2	Thermal stresses in a composite cylinder.	242
9.5.3	Transient thermal stresses in a thin plate.	245

10. Elastic stability	249
10.1 Introduction	249
10.2 Direct and energy methods	250
10.3 Euler's method for axially compressed columns	252
10.3.1 Critical buckling load	252
10.3.2 Critical buckling stress	254
10.3.3 Remarks	256
10.4 Energy-based approximate method	256
10.5 Non-conservative loads	258
10.6 Solved problems	259
10.6.1 Two rigid bars connected with a spring	259
10.6.2 Column clamped at one end	260
10.6.3 Column clamped at one end and simply supported at the other	261
10.6.4 Column clamped at both ends	263
10.6.5 Column elastically built-in at one end and simply sup- ported at the other	264
10.6.6 Column with non-uniform properties	266
10.6.7 Eccentric compressive load	267
10.6.8 Beam-column under compressive and bending forces ..	268
10.6.9 Energy method	269
11. Theory of thin shells	273
11.1 Geometry of the mid-surface	273
11.2 First fundamental form	274
11.3 Second fundamental form	278
11.4 Compatibility conditions of Codazzi and Gauss	280
11.5 Surface of revolution	280
11.5.1 General case	280
11.5.2 Conic surfaces	283
11.6 Gradient of a vector field in curvilinear coordinates	283
11.7 Kinematics of the mid-surface	285
11.8 Displacements and strains outside the mid-surface	287
11.8.1 General theory	287
11.8.2 Application: plates in rectangular coordinates	289
11.8.3 Application: plates in polar coordinates	290
11.9 Internal loads (stress resultants)	290
11.10 Equilibrium equations	292
11.10.1 General theory	292
11.10.2 Application: plates in rectangular coordinates	295
11.10.3 Application: plates in polar coordinates	295
11.11 Constitutive equations	296
11.12 Membrane theory	298
11.13 Further reading	298

12. Elasto-plasticity	301
12.1 One-dimensional model	301
12.2 Three-dimensional model	304
12.3 Linear elasticity	305
12.4 Equivalent stress	306
12.5 Hardening	306
12.6 Flow rules	307
12.7 Tangent operator, loading/unloading, hardening/softening . . .	308
12.8 Elementary examples	312
12.8.1 Uniaxial tension-compression	312
12.8.2 Simple shear	313
12.9 Boundary-value problem	313
12.10 Numerical algorithms	313
12.10.1 Finite element method (F.E.M.)	313
12.10.2 Return mapping algorithm	315
12.10.3 Consistent tangent operator	318
12.11 A general framework for material models	320
12.11.1 State variables	320
12.11.2 Equations of state	321
12.11.3 Flow rules	322
12.11.4 Rate-independent plasticity	323
12.11.5 Heat equation	324
12.12 A class of non-associative plasticity models	325
12.13 Further reading	327
13. Elasto-viscoplasticity	329
13.1 One-dimensional model	329
13.2 Three-dimensional model	331
13.3 Numerical algorithms	332
13.3.1 Return mapping algorithm	333
13.3.2 Consistent tangent operator	333
13.4 Further reading	335
14. Nonlinear continuum mechanics	337
14.1 Kinematics	337
14.1.1 Description of motion	338
14.1.2 Material time derivative	338
14.2 Deformation	339
14.2.1 Deformation gradient	339
14.2.2 Polar decomposition	340
14.2.3 Spectral decompositions	341
14.2.4 Length variation	343
14.3 Strain measures	344
14.3.1 One-dimensional case	344
14.3.2 Three-dimensional case	344

14.4	Strain rates	346
14.5	Balance laws	348
14.5.1	Transport formula	348
14.5.2	Conservation of mass	349
14.5.3	Conservation of linear momentum	350
14.5.4	Conservation of rotational momentum	350
14.5.5	Cauchy stress tensor	351
14.5.6	Eulerian strong formulation	351
14.5.7	Eulerian weak formulation	352
14.5.8	Balance of work and energy rates	353
14.5.9	Nominal stress	354
14.5.10	Lagrangian weak formulation	354
14.5.11	Lagrangian strong formulation	355
14.6	Conjugate stress and strain measures	356
14.6.1	Definition and examples	356
14.6.2	Interpretation of the second Piola-Kirchhoff stress	358
14.6.3	Uniaxial tension/compression	358
14.7	Objectivity	359
14.7.1	Definition	360
14.7.2	Examples	361
14.8	Objective stress rates	363
14.8.1	Examples	363
14.8.2	A family of objective rates	365
14.9	Laws of thermodynamics	366
14.9.1	First law	366
14.9.2	Second law	366
14.9.3	Clausius-Duhem inequality	367
14.10	Further reading	367
15.	Nonlinear elasticity	369
15.1	Hyperelasticity and hypoelasticity	369
15.1.1	Definitions	369
15.1.2	Hyperelasticity and material objectivity	369
15.1.3	Elasticity tensors	371
15.1.4	Incompressibility constraint	373
15.2	Principal invariants and principal stretches	374
15.3	Isotropic hyperelasticity in principal invariants	375
15.3.1	Formulation	375
15.3.2	Modified neo-Hookean model	377
15.3.3	Modified Mooney-Rivlin model	378
15.4	Isotropic hyperelasticity in principal stretches	379
15.4.1	Formulation	379
15.4.2	Modified Ogden's model	380
15.5	Examples of homogeneous deformations	381
15.5.1	Homogeneous simple shear	381

15.5.2	Uniform extension	383
15.5.3	Pure dilatation	384
15.6	Linearization	385
15.6.1	Linearization of the deformation	386
15.6.2	Linearization of constitutive equations	387
15.6.3	Linearization of the equations of elasto-statics	387
15.6.4	Variational formulations	388
15.6.5	Linearization of the weak formulation	389
15.7	Mixed variational formulation	390
15.7.1	Formulation	390
15.7.2	Incompressibility constraint	392
15.8	Appendices	393
15.8.1	The Piola identity	393
15.8.2	Linearization of a pressure B.C.	393
15.8.3	Differentiation of an isotropic function of a second-order symmetric tensor	394
15.8.4	Elasticity tensors for principal stretch formulation	395
16.	Finite-strain elasto-plasticity	397
16.1	First theory	397
16.1.1	Multiplicative decomposition of the deformation gradient	397
16.1.2	Hyperelastic-plastic constitutive equations	399
16.1.3	Stress-strain relations	401
16.1.4	Flow rules	402
16.1.5	Elastic predictor	404
16.1.6	Time discretization of the plastic flow rule	404
16.1.7	Return mapping algorithm in principal stresses and strains	406
16.1.8	Algorithmic tangent moduli	407
16.1.9	Summary of the algorithm	408
16.1.10	Application: Quadratic logarithmic free energy and J2 flow	410
16.2	Second theory	413
16.2.1	Additive decomposition of the rate of deformation and hypoelasticity	413
16.2.2	Computation of the strain increment	415
16.2.3	Polar decomposition algorithm	417
16.2.4	A time-integration algorithm for the rotation matrix	419
16.2.5	Application: the Jaumann objective stress rate	420
16.2.6	Summary	421
16.2.7	Incremental objectivity	422

17. Cyclic plasticity	423
17.1 One-dimensional model	423
17.2 Three-dimensional model	425
17.3 Dissipation inequality	428
17.4 Plastic multiplier	428
17.5 Tangent operator	429
17.6 Hardening modulus	429
17.7 Return mapping algorithm	430
17.8 Consistent tangent operator	433
17.9 Numerical simulation	435
18. Damage mechanics	439
18.1 Damage variable	439
18.2 Three-dimensional constitutive model	442
18.3 Dissipation inequality	445
18.4 Plastic multiplier	445
18.5 Tangent operator	445
18.6 Hardening modulus	446
18.7 Closed-form solutions for loadings with constant triaxiality . .	447
18.8 Return mapping algorithm	450
18.8.1 Corrections over the elastic predictor	458
18.8.2 Summary of the algorithm	458
18.8.3 Non-damaged case	458
18.9 Consistent tangent operator	459
18.9.1 Non-damaged case	461
18.10 Numerical simulations	461
18.10.1 Ductile failure under uniaxial tension	462
18.10.2 Ductile failure under simple shear	462
18.10.3A post-processor for crack initiation	462
18.11 Further reading	465
19. Strain localization	469
19.1 Motivation: a one-dimensional example	469
19.2 Uniqueness and ellipticity	472
19.3 Strain localization	474
19.3.1 Continuous bifurcation	475
19.3.2 Discontinuous bifurcation	475
19.3.3 Summary	477
19.4 Analytical results for initially homogeneous plane problems . .	477
19.4.1 General strain-softening models	477
19.4.2 A ductile damage model	481
19.5 Numerical results for a ductile damage model	481
19.5.1 Biaxial loadings in plane stress	481
19.5.2 Notched plate with a macro-defect	484
19.6 Nonlocal or internal-length models	496

XVIII Table of Contents

19.7	A two-scale homogenization procedure	498
19.8	Numerical algorithms.	503
19.9	Elasticity with damage- Model without threshold	506
19.9.1	Local constitutive equations	506
19.9.2	Nonlocal macroscopic formulation	506
19.9.3	Numerical simulations.	507
19.10	Elasticity with damage- Model with threshold	509
19.10.1	Local constitutive equations.	509
19.10.2	Nonlocal macroscopic formulation	510
19.10.3	Numerical simulations.	510
19.11	Appendices.	513
19.11.1	Strain localization criterion in 2D.	513
19.11.2	Macroscopic free-energy potential	515
19.11.3	Macroscopic dissipation potential	517
20.	Micro-mechanics of materials.	519
20.1	Micro/macro approach	519
20.2	Homogenization schemes.	521
20.2.1	Average strains and stresses.	521
20.2.2	Voigt model	523
20.2.3	Reuss model	524
20.2.4	Self-Consistent model	524
20.2.5	Mori-Tanaka model	526
20.3	Micro/macro constitutive model for semi-crystalline polymers	527
20.3.1	Crystalline phase	528
20.3.2	Amorphous phase	531
20.3.3	Intermediate phase	536
20.3.4	Single inclusion	538
20.3.5	Overall behavior.	538
20.3.6	Numerical simulations.	538
20.4	Further reading	543
A.	Cylindrical coordinates	545
B.	Cardan's formulae	551
C.	Matrices for the representation of second- and fourth-order tensors	553
C.1	Storage.	553
C.2	Change of coordinates.	556
D.	Zero-stress constraints	559
D.1	Small-strain J_2 elasto-plasticity.	559
D.2	General small-strain models	561
D.3	General finite-strain models	562