Contents

Part I Basics

1	Basi	cs About	Graphs	3
	1.1	The La	inguage of Graphs	3
		1.1.1	Connectedness and the Distance Metric	4
		1.1.2	Subgraphs	5
		1.1.3	Special Types of Graphs and Subgraphs	6
		1.1.4	Metrical Properties of Subgraphs	9
	1.2	Morph	isms of Graphs	14
		1.2.1	Basic Definitions	15
		1.2.2	Fiberings, Covering Morphisms, and Lifts of Walks	16
		1.2.3	Universal C-Covers	18
	1.3	C-Hom	otopy	19
		1.3.1	Further Properties of the Closure Operator on Circuits	21
		1.3.2	Control of \hat{C} -Connectedness Through a Subgraph	24
		1.3.3	Tits' Condition for Being Simply C-Connected	26
	1.4	The Ex	tistence of Universal C-Covers	31
		1.4.1	C-Connectedness and Subgraphs	33
		1.4.2	The Construction	34
		1.4.3	Deck Transformations	36
	1.5	Exercis	ses for Chapter 1	37
		1.5.1	Exercises for Section 1.1	
		1.5.2	Exercises for Section 1.2	40
		1.5.3	Exercises for Section 1.3	41
2	Geor	metries:	Basic Concepts	43
	2.1		ection	43
	2.2	Geome	tries: Definitions and Basic Concepts	
		2.2.1	Basic Definitions	43
		2.2.2	Subgeometries	44
		2.2.3	Truncations	44
		2.2.4	Point-Line Geometries	45

хi

xii Contents

		2.2.5 Flags and Chambers	45
		2.2.6 Residues	45
		2.2.7 The Interplay of Residues and Truncations	46
		2.2.8 Shadows	46
	2.3	Examples	47
	2.4	Morphisms of Geometries	50
		2.4.1 Definition	50
		2.4.2 Automorphisms	50
		2.4.3 Morphisms Defined by a Group of Automorphisms	52
		2.4.4 Truncations and Morphisms	53
		2.4.5 Residues and Morphisms	53
	2.5	Connectedness Properties	54
		2.5.1 Residual Connectedness	54
	2.6	Exercises for Chapter 2	57
		•	
•	n · .	T. C	5 0
3		Line Geometries	59
	3.1	Introduction	59
		3.1.1 On Choosing a Reasonable Definition	59
		3.1.2 Our Definition of Point-Line Geometry	60
	3.2	The Point-Collinearity Graph	61
	3.3	Morphisms and Covers of Point-Line Geometries	61
	3.4	Subspaces	62
		3.4.1 Generalizations of the Notion of Subspace	65
	3.5	Special Types of Point-Line Geometries	66
		3.5.1 Partial Linear Spaces and Linear Spaces	66
		3.5.2 Gamma Spaces	66
	3.6	Local Connectedness in Gamma Spaces	67
		3.6.1 The Decomposition of a Gamma Space into Locally	
		Connected Components	68
		3.6.2 How Local Characterizations of Gamma Spaces Reduce	
		to the Locally Connected Case	69
	3.7	Enriching Geometries	70
	3.8	Products of Point-Line Geometries, a Construction	71
	3.9	Exercises and Examples for Chapter 3	72
4	Hype	erplanes, Embeddings, and Teirlinck's Theory	79
	4.1	Veldkamp Spaces	79
		4.1.1 Introduction	79
		4.1.2 Geometric Hyperplanes	80
		4.1.3 The Veldkamp Space	81
	4.2	Teirlinck's Theory	82
		4.2.1 The Hypotheses.	82
		4.2.2 The Exchange Property in Closed Sets	83

Contents xiii

		4.2.3	To What Extent Does H Separate Points?	84
		4.2.4	The \mathcal{H} -Closure of Two Inequivalent Points	
		4.2.5	The Natural Morphism	86
		4.2.6	Singular Subspaces are Generalized Projective Spaces	87
	4.3	The Ef	fect of Teirlinck's Theory on the Veldkamp Space	89
		4.3.1	The Veblen Axiom	89
		4.3.2	The First Reason Why a Veldkamp Space Might	
			be a Projective Space	91
		4.3.3	Another Reason why the Veldkamp Space $\mathcal{V}_{\mathcal{H}}$ Might	
			be a Projective Space	93
	4.4	The Co	onnections Between Projective Embeddings	
			eldkamp Spaces	94
	4.5	Teirline	ck's Theory and Embedability into the Second	
			mp Space	96
	4.6	Exercis	ses for Chapter 4	
		4.6.1	Exercises on Geometric Hyperplanes	
		4.6.2	Exercises Relating to Veldkamp Spaces	
		4.6.3	Exercises Relating to Teirlinck's Axioms	101
5	Proj	jective Pl	lanes	105
_	5.1		action	
	5.2		xioms for a Projective Plane	
	5.3		Planes	
	5.4	The Re	elation Between Affine and Projective Planes	111
	5.5	Transla	ation Planes	112
		5.5.1	Introduction	112
		5.5.2	Translation Planes from Groups	
		5.5.3	Endomorphism Rings	
		5.5.4	The Kernel of (G, S)	
				118
		5.5.5	A Short Course on the Klein Correspondence	
		5.5.6	Ovoids and Spreads	119
	5.6	5.5.6 Recogn	Ovoids and Spreads	119 120
	5.6	5.5.6 Recogn 5.6.1	Ovoids and Spreads	119 120 120
	5.6	5.5.6 Recogn 5.6.1 5.6.2	Ovoids and Spreads	119 120 120 123
	5.6	5.5.6 Recogn 5.6.1 5.6.2 5.6.3	Ovoids and Spreads nizing the Classical Planes Perspective Collineations The Moufang Case The Desarguesian Case	119 120 120 123 124
		5.5.6 Recogn 5.6.1 5.6.2 5.6.3 5.6.4	Ovoids and Spreads nizing the Classical Planes Perspective Collineations The Moufang Case The Desarguesian Case The Proof of Theorem 5.6.8	119 120 120 123 124 126
	5.65.7	5.5.6 Recogn 5.6.1 5.6.2 5.6.3 5.6.4 Append	Ovoids and Spreads nizing the Classical Planes Perspective Collineations The Moufang Case The Desarguesian Case The Proof of Theorem 5.6.8 dix to Chapter 5	119 120 120 123 124 126 127
		5.5.6 Recogn 5.6.1 5.6.2 5.6.3 5.6.4	Ovoids and Spreads nizing the Classical Planes Perspective Collineations The Moufang Case The Desarguesian Case The Proof of Theorem 5.6.8	119 120 120 123 124 126 127

xiv Contents

6	Proje	ective Sp	aces	. 135
	6.1	Introdu	ction	. 135
	6.2	Basic P	Properties of Projective Spaces and Their Partially Ordered	
			s of Subspaces	. 136
		6.2.1	The Definition of a Projective Space	. 136
		6.2.2	Matroids	
		6.2.3	The Matroid of a Projective Space	
	6.3	Subplai	nes of Larger Projective Spaces	
	6.4		ding Three-dimensional Projective Spaces	
		6.4.1	Affine Spaces	. 143
		6.4.2	The Classical Affine Space $AG(n, D)$. 145
		6.4.3	Completing an Arbitrary Affine Space to its Associated	
			Projective Space	. 147
	6.5		ding Three-dimensional Projective Spaces,	
		Some of	of Whose Planes are Classical	
		6.5.1	Introduction	
		6.5.2	Synthesizing Coordinates: A Basic Lemma	
		6.5.3	Coordinatizing <i>P</i> Minus a Desarguesian Plane	
	6.6	Extendi	ing Projective Embeddings of Projective Spaces	. 153
	6.7		ain Theorem: All Projective Spaces of Projective	
		Dimens	sion at Least Three are Classical	. 155
	6.8	The Fu	ndamental Theorem of Projective Geometry	. 158
	6.9	Three T	Technical Results Concerning Projective Spaces	. 160
	6.10		es for Chapter 6	
		6.10.1	Exercise on the Definition of Projective Space	. 163
		6.10.2	Axioms of Affine Space	
		6.10.3	Simple Results on Embedded Projective Spaces	
7	Polar	·Snaces		167
•	7.1		ction	
	7.2		lized Quadrangles	
	7.3		of Polar Spaces	
	,	7.3.1	Definition	
		7.3.2	Opposite Lines and the Radical	
		7.3.3	The Basic Morphism	
		7.3.4	Basic Properties of Non-degenerate Polar Spaces	
		7.3.4	The Rank of a Polar Space	
	7.4		e Polar Spaces of Rank Three	
	7.4		Properties of Abstract Polar Spaces	
	1.5	7.5.1	Geometric Hyperplanes of Polar Spaces	
		7.5.1 7.5.2		. 102
		1.3.2	Hyperplane Complements in Polar Spaces	104
		7.50	of Rank at Least Three	
		7.5.3	Automorphisms of Non-degenerate Polar Spaces	. 186

Contents xv

	7.0	The Cla	issical Polar Spaces	. 192
		7.6.1	Morphisms from a Projective Space to Its Dual,	
			or Where Sesquilinear Forms Come From	
		7.6.2	Quasi-Polarities	. 195
		7.6.3	The Relation Between Quasi-Polarities	
			and Reflexive Forms	. 197
		7.6.4	Admissible Pairs and Gramm Matrices	. 202
		7.6.5	PseudoQuadratic Forms	. 207
		7.6.6	The Polar Space of a Pseudoquadratic Form	. 209
	7.7	Polar S	paces Embedded in Projective Spaces	. 211
		7.7.1	Introduction	. 211
		7.7.2	The Geometry of Embedded Polar Spaces	. 212
		7.7.3	The Main Theorem	. 217
	7.8	Polar S	paces Embedded in a Classical Polar Space	. 218
	7.9	Non-de	generate Polar Spaces of Rank at Least Four	
			issical	. 225
		7.9.1	Introduction	. 225
		7.9.2	Teirlinck's Condition in V_p	. 226
		7.9.3	The Classification of Non-degenerate Polar Spaces	
			of Rank at Least Four	. 230
	7.10	Epilogu	ıe	
		7.10.1	Non-embeddable Rank Three Polar Spaces	
		7.10.2	Tits' Classification Theorem	
		7.10.3	The Buekenhout-Johnson-Shult Theorem	. 236
	7.11	Exercise	es for Chapter 7	. 237
		7.11.1	Exercises Involving Only Basic Properties	
		7.11.2	Exercises on Classical Polar Spaces	
		7.11.3	Exercises from Section 7.9	
	7.12	Append	lix to Chapter 7: Known Quadrangles in 2001	
		7.12.1	The Older Examples	
		7.12.2	The Last 15 Years Before 2001: Kantor-Families,	
			the Heisenberg Group and Flocks of a Quadratic Cone	. 246
		7.12.3	Prospects for a Classification	
8	Near	Polygon	s	. 251
	8.1		ction	
		8.1.1	What We Know About Polar Spaces	
		8.1.2	The Definition of Near Polygons	
		8.1.3	Some Non-classical Examples	
		8.1.4	Near Polygons Arising from Chamber Systems	
			of Buildings	. 257
		8.1.5	Near Polygons of Fischer Type	257
		8.1.6	Classical Near Polygons: The Dual Polar Spaces	
		0.1.0	Diamotoni rioni i diggondi, riio Duni i dini dencos, riirii	

xvi Contents

	8.2	The Ex	istence of Quads in Near Polygons	
		8.2.1	Internal and External Diameter	
		8.2.2	Bounded Diameter Point-Sets in Near Polygons	262
		8.2.3	The Existence of Quads	263
	8.3	The Co	onvexity of Quads	264
	8.4		cometry of Quads	
		8.4.1	The Diagram	266
		8.4.2	The Relations Between Quads	267
		8.4.3	The Relations Among Points and Quads	
	8.5	A Chai	racterization of Dual Polar Spaces	
		8.5.1	The Axioms	
		8.5.2	The Structure of the Residue of a Point	270
		8.5.3	The (Global) Convex Subspace Structure	273
		8.5.4	The Dual Polar Space Structure in Near Polygons	
			of Finite Diameter	279
		8.5.5	Recapitulation	
	8.6	Exercis	ses for Chapter 8	
			-	
D		3.5.41	1 1	
Par	t III	Method	1010gy	
9	Cha	mbar Cr	stems and Buildings	201
9	9.1		ection	
	9.1		per Systems	
	9.2	9.2.1	The Chamber System of a Geometry	
		9.2.1	Abstract Chamber Systems	
		9.2.2		
	0.2		Residually Connected Chamber Systems	
	9.3	9.3.1	per Systems with Strongly Gated Residues	
		9.3.1		
			Basic Properties Concerning Strongly Gated Residues	
		9.3.3	Intersections of Strongly Gated Residues	312
		9.3.4	2-Simply Connectedness is a Consequence of Strong	214
		025	Gatedness at Low Rank	
		9.3.5	Λ-Homotopy	
		9.3.6	Further Consequences of the Hypothesis (RG ₂)	
	0.4	9.3.7	Equivalence of Various Gatedness Conditions	
	9.4		alized Polygons	
		9.4.1	Panel Homotopy	
		9.4.2	The Chamber System of a Generalized Polygon	
		9.4.3	Generalized <i>n</i> -Gons as Geometries	
		9.4.4	Existence of Generalized Polygons	
	9.5	_	ms	
		9.5.1	Introduction	
		9.5.2	Rank Two Diagrams	
		9.5.3	Diagram Geometries of Higher Rank	347

Contents xvii

		9.5.4	Chamber Systems Belonging to a Diagram	348
		9.5.5	Diagrams and the Functors Connecting Chamber	
			Systems and Geometries	349
		9.5.6	Some Examples Concerning Diagram Geometries	
			and Chamber Systems	349
	9.6	Chamb	er Systems with a Coxeter Diagram	350
		9.6.1	Coxeter Groups and Coxeter Systems	
		9.6.2	The Cayley Graph of the Coxeter System (W, R, M) ,	
			and the Coxeter Chamber Systems	353
		9.6.3	Other Properties of Coxeter Chamber Systems	
		9.6.4	Walls, Roots, and Distance in a Coxeter	
			Chamber System	363
		9.6.5	Gatedness and Convexity of Residues	
		9.6.6	When Is a Coxeter Group Finite?	
	9.7	Chamb	er Systems of Type M	
		9.7.1	Introduction	
		9.7.2	The Three Levels of Homotopy	
	9.8	Buildin	ıgs	
		9.8.1	Introduction	
		9.8.2	The Conditions (G_c) and (P_c)	
	9.9	Apartm	nents	
		9.9.1	The Tits Metric of a Building	
		9.9.2	Strong Isometries and the Standard Apartment Axioms	
			for a Building	386
	9.10	Append	dix to Chapter 9: Spherical Buildings and (B, N) -Pairs	
		9.10.1	Tits Systems	
		9.10.2	(B, N)-Pairs and Tits Systems	
		9.10.3	Sphericity	
	9.11	-	ses for Chapter 9	
		9.11.1	Exercises on Chamber Systems	
		9.11.2	Exercises on Residual Connectedness	
		9.11.3	A Few Exercises on Gatedness	
		9.11.4	Exercises on Generalized Polygons	
		9.11.5	An Interplay of Examples and Exercises	
		J.111.0	on Diagram Geometries	395
		9.11.6	Exercises Concerning Chamber Systems of Type M	
		2.11.0	Exercises concerning channel systems of Type in	
10	2-Co	vers of (Chamber Systems	399
	10.1		ction	
	10.2	Kasiko	va's Lemma on 2-Coverings of Chamber Systems	401
	10.3		portant Property of Buildings	
	10.4		Are Universal 2-Covers Buildings?	
	10.5		ructure of an M-Homotopy Class	
	10.6	Tits' "I	Local Approach" Theorem	412

xviii Contents

11	Loca	lly Truncated Diagram Geometries	. 415			
	11.1	Locally Truncated Geometries: A Quick Review	. 415			
	11.2	The Idea of Sheaves	. 416			
		11.2.1 Flags and Residues	. 416			
		11.2.2 Sheaves	. 417			
		11.2.3 The Chamber System Associated with a Sheaf	. 417			
	11.3	Sheaves for Locally Truncated Geometries	. 419			
		11.3.1 Modelling Geometries on Truncations	. 419			
		11.3.2 Geometric Rigidity				
		11.3.3 Constructing Sheaves				
	11.4	Recovering Γ from $C(\mathcal{F})$				
	11.5	Residual Connectedness of Γ and $(C(\mathcal{F}))_J$. 429			
	11.6	Configurations Produced by a 1-Covering of $C(\mathcal{F})$. 431			
	11.7	Applying the Local Approach Theorem: A Useful Tool				
		for Point-Line Characterizations	. 433			
	11.8	A Covering of a Point-Collinearity Graph Derived from a Locally				
		Truncated Geometry	. 434			
	11.9	Exercises for Chapter 11				
		11.9.1 Exercises Involving Connectedness Properties	. 437			
		11.9.2 Morphisms by Chamber-Semiregular Automorphism				
		Groups Do Not Preserve Residual Connectedness	. 438			
12	Sono	rated Systems of Singular Spaces	441			
14	12.1	Introduction				
	12.1	12.1.1 The Basic Context: Paraprojective Spaces				
		12.1.2 Local and Global Hypotheses				
		12.1.3 Separated Systems of Singular Subspaces				
	12.2		. 44.7			
	12.2 Geometries with Two Systems of Subspaces Meeting at Lines					
		12.2.1 Introduction				
		12.2.2 Geometries with Two Systems of Subgeometries				
		12.2.3 A Purely Local View				
	12.3	A Criterion for Global Separation of Singular Subspaces				
	12.5	12.3.1 Examples Illustrating Local Separation	. 447			
		Without Global Separation	117			
		12.3.2 A Sufficient Condition for Local Separation	. 44)			
		to Imply Global Separation	450			
	12.4	Exercises for Chapter 12	450			
	12.4	Exercises for Chapter 12	. 432			
13		perstein's Theory of Symplecta and Parapolar Spaces				
	13.1	Introduction				
	13.2	Polar Families				
	13.3	Elementary Properties of Symplecta	. 461			
	13.4	Parapolar Spaces: A Stage for Characterizing				
		the Lie Incidence Geometries	. 462			

Contents xix

		13.4.1	The Definition of Parapolar Spaces	462
		13.4.2	Examples of Parapolar Spaces	465
	13.5	Local C	Connectedness in Parapolar Spaces	
		13.5.1	Introduction	
		13.5.2	Isolating Polar Spaces from Locally Connected	
			Parapolar Spaces	471
	13.6	Hanssei	ns' Principle	472
	13.7	When I	s Either $\Delta_2^*(p)$ or $P_2^*(p)$ a Subspace	
		of a Par	rapolar Space?	478
	13.8	General	lized Parapolar Spaces	482
		13.8.1		
		13.8.2	Generalized Parapolar Spaces of Nexus 1	486
		13.8.3	Generalized Parapolar Spaces of Higher Nexus	487
	13.9	Exercis	es for Chapter 13	488
		13.9.1	Exercises Concerning Parapolar Spaces	488
		13.9.2	Exercises Concerning an Application	
			of Hanssens' Principle	
		13.9.3	Exercises for Section 13.7	493
		13.9.4	Exercises for Section 13.8: On Generalized Parapolar	
			Spaces of Higher Nexus	493
		13.9.5	Symplecta Which Cannot Intersect at a Single Point	493
Pai	rt IV	Applica	tions to Other Lie Incidence Geometries	
14	Chai	racteriza	tions of the Classical Grassmann Spaces	497
	14.1		st Characterization	
		14.1.1	Axioms on Two Families of Maximal	
			Singular Subspaces	497
	14.2	A Varia	ation for Half-Grassmann Spaces	
		14.2.1	Consequences of Axioms (G1) and (G2)	
		14.2.2	The Effect of the Other Axioms	
	14.3	Sprague	e's Theory	
		14.3.1	Introduction	
		14.3.2		
			(G1)-Geometries with the Veblen and Dual-Veblen	511
				511
		14.3.2	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511
	14.4	14.3.2 14.3.3	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519
15		14.3.2 14.3.3 Exercise	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519
15	Char	14.3.2 14.3.3 Exercise	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519 523
15	Char The	14.3.2 14.3.3 Exercise acterizin Cohen-C	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519 523
15	Char The (14.3.2 14.3.3 Exercise acterizin Cohen-C Prognos	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519 523 527 527
15	Char The	14.3.2 14.3.3 Exercise acterizin Cohen-C Prognose Introduce	(G1)-Geometries with the Veblen and Dual-Veblen Axioms, (G2) and (H3)	511 513 519 523 527 527 528

xx Contents

		15.3.1	A Technical Result	530
		15.3.2	Cohen's Hypotheses and Their Elementary	
			Consequences	530
	15.4	Strong	Parapolar Spaces with Constant Symplectic Rank $r > 3$	
		and Sat	isfying $(CC)_{r-2}$	535
		15.4.1	Introduction	535
		15.4.2	The Case $r = 4 \dots$	536
		15.4.3	The Case $r = 5$	537
		15.4.4	The Case $r = 6 \dots$	538
		15.4.5	The Case $r > 6$	540
		15.4.6	The First Cohen–Cooperstein Theorem	540
	15.5	The Un	iformizing Condition	541
		15.5.1	Introduction	541
		15.5.2	A Basic Lemma	541
		15.5.3	The Principle	542
		15.5.4	The Uniformizing Principle and Constant	
			Symplectic Rank	543
	15.6	Exercis	es for Chapter 15	545
		15.6.1	General Features	545
		15.6.2	Understanding the Exceptional Strong	
			Parapolar Spaces	545
		15.6.3	Involutions Acting on Classical Strong	
			Parapolar Spaces	5/10
			Parapolar Spaces	340
			rarapolar Spaces	346
				540
16			ng Strong Parapolar Spaces by the Relation Between	
16	Point	ts and Co	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553
16	Point 16.1	ts and Co Introdu	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553
16	Point	ts and Co Introdu Parapol	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553 553
16	Point 16.1	ts and Co Introduce Parapol 16.2.1	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553 553 554
16	Point 16.1 16.2	Is and Co Introduce Parapol 16.2.1 16.2.2	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553 553 554 555
16	Point 16.1	Is and Co Introduce Parapole 16.2.1 16.2.2 The Sec	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553 553 554 555 557
16	Point 16.1 16.2	Is and Co Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces	553 553 554 555 557
16	Point 16.1 16.2	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis" Elementary Consequences of the "Clean Hypothesis" The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta	553 553 554 555 557 558
16	Point 16.1 16.2	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis" Elementary Consequences of the "Clean Hypothesis" The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces	553 553 554 555 557 558 558
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2)	553 554 555 557 558 560 561
16	Point 16.1 16.2	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) al Case 1: d = 0	553 554 555 557 558 560 561 563
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) all Case 1: d = 0 Fibered Symplecta	553 554 555 557 558 560 561 563
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) all Case 1: d = 0 Fibered Symplecta Unfibered Symplecta: M-Projections which Embed	553 553 554 555 557 558 560 561 563
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin 16.4.1 16.4.2	ng Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis" Elementary Consequences of the "Clean Hypothesis" The Third Case of the Trichotomy cond Case of the Trichotomy The Maximal Singular Subspaces The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) all Case 1: d = 0 Fibered Symplecta Unfibered Symplecta: M-Projections which Embed a Symplecton	553 553 554 555 558 560 561 563 564 565
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin 16.4.1 16.4.2	Ing Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy cond Case of the Trichotomy. The Maximal Singular Subspaces. The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) Tall Case 1: $d = 0$ Fibered Symplecta Unfibered Symplecta: M -Projections which Embed a Symplecton The Case that Some Member of \mathcal{M} is a Line	553 553 554 555 558 560 561 563 564 565
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin 16.4.1 16.4.2 16.4.3 16.4.4	Ing Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy. cond Case of the Trichotomy. The Maximal Singular Subspaces. The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) and Case 1: $d = 0$ Fibered Symplecta Unfibered Symplecta: M -Projections which Embed a Symplecton. The Case that Some Member of \mathcal{M} is a Line Symplecta Disjoint from No Member of \mathcal{M}	553 553 554 555 558 560 561 563 565 567 569
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin 16.4.1 16.4.2	Ing Strong Parapolar Spaces by the Relation Between Parain Maximal Singular Subspaces Ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy. The Maximal Singular Subspaces. The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) Tal Case 1: $d = 0$ Fibered Symplecta Unfibered Symplecta: M -Projections which Embed a Symplecton The Case that Some Member of \mathcal{M} is a Line Symplecta Disjoint from No Member of \mathcal{M} When All Symplecta are Grids	553 554 555 557 558 560 561 563 565 565
16	Point 16.1 16.2 16.3	Introduce Parapol 16.2.1 16.2.2 The Sec 16.3.1 16.3.2 16.3.3 16.3.4 The Fin 16.4.1 16.4.2 16.4.3 16.4.4 16.4.5 16.4.6	Ing Strong Parapolar Spaces by the Relation Between ertain Maximal Singular Subspaces ction ar Spaces with the "Clean Hypothesis". Elementary Consequences of the "Clean Hypothesis". The Third Case of the Trichotomy. cond Case of the Trichotomy. The Maximal Singular Subspaces. The Symplecta More Maximal Singular Subspaces The Final Result for Case 2 with (H2) and Case 1: $d = 0$ Fibered Symplecta Unfibered Symplecta: M -Projections which Embed a Symplecton. The Case that Some Member of \mathcal{M} is a Line Symplecta Disjoint from No Member of \mathcal{M}	553 553 554 555 557 568 563 565 567 569 571

Contents xxi

	16.6	A Coro	llary of Theorem 16.2.1	. 575
		16.6.1	Uniformity of Point-Residuals	. 576
		16.6.2	The Proof of Corollary 16.6.1	. 578
	16.7	A Furth	er Corollary of Theorem 16.2.1	. 581
	16.8	Append	lix to Chapter 16	. 596
		16.8.1	Background	. 596
		16.8.2	Admissible Triples	. 596
		16.8.3	Constructions of the Relevant Parapolar Spaces	. 598
17	Point	t-Line Cl	naracterizations of the "Long Root Geometries"	. 603
	17.1	Introduc	ction	
		17.1.1	Long Root Geometries	
		17.1.2	The Main Results of this Chapter	
	17.2		ial Class of Strong Parapolar Spaces	
	17.3	The Pro	oof of Theorem 17.1.1	. 617
		17.3.1	Simple-connectedness of the Point-Collinearity	
			Graph of Γ	
		17.3.2	The Uniform Structure of the Point-Residuals	
		17.3.3	The Case of Finite Singular Rank	
		17.3.4	The Case of Infinite Singular Rank	
	17.4		es for Chapter 17	
	17.5	Append	lix to Chapter 17: Long Root Geometries Updated	. 626
18	The l	Peculiar	Pentagon Property	. 627
	18.1	Introdu	ction	. 627
		18.1.1	The Pentagon Axiom; Its Beginnings	. 628
		18.1.2	Where Are We Going with this Axiom?	
		18.1.3	Introduction	. 629
	18.2	Strong 1	Parapolar Spaces with the Pentagon Property	
		18.2.1	Examples	. 630
		18.2.2	The Role of Grids	
		18.2.3	Why We Assume that Γ Is Not a Polar Space	. 631
		18.2.4	What Extra Axioms Are Needed?	. 632
	18.3	Classify	ying the Parapolar Spaces Satisfying (PL) and (PL*)	
		18.3.1	An Elementary Consequence of (PL*)	. 633
		18.3.2	The Case that \mathcal{D} is Empty and a Revisitation	
			of Cohen's Theorem	634
		18.3.3	The Case that Some Symplecton has Polar Rank	
			at Least Three	637
		18.3.4	The Case that $\mathcal{D} \neq \emptyset$ and All Members of \mathcal{Q} Have Polar	
			Rank Two	640
		18.3.5	What We Have Proved	
	18.4	The Pro	oof of the Main Theorem	. 647
		1841	The Hypothesis (H)	64

xxii Contents

		18.4.2	The Axiom (Q*)	650
		18.4.3	Local Uniformity	650
		18.4.4	The Metasymplectic Case	652
		18.4.5	Beyond the Metasymplectic Case: A Related Geometry	652
		18.4.6	Regarding Γ^* as a Locally Truncated Geometry	653
		18.4.7	Enriching Γ^* to a Rank Four Geometry via	
			Hanssen's Principle	. 654
		18.4.8	211 - 211 - 211 - 21 - 21 - 21 - 21 - 2	
			of a Polar Grassmannian	. 654
		18.4.9	The Simple Connectedness of the Geometry Γ in the	
			Case that $k = 2$	
	18.5		omments on the Main Theorem	
	18.6	Exercise	es for Chapter 18	657
		18.6.1	Relaxing Hypothesis (PL*)	
		18.6.2	Relaxing (Q): What if the Class Q is Empty?	659
Re	ference	es		665
Ind	lov			671