

Contents

Preface — VII

1	Motivation of the Hardy–Leray potential — 1
1.1	Introduction — 1
1.2	Where does the monster appear? — 2
1.2.1	The Hardy–Leray potential in quantum mechanics — 2
1.2.2	More quantum mechanics: the uncertainty principle — 4
1.2.3	Frank-Kamenetskii model in combustion: a Gelfand problem — 5
1.2.4	Stability of the singular solution — 12
1.3	A Picone type inequality — 12
2	Looking at the Hardy–Leray potential — 15
2.1	Properties of the Hardy–Leray potential — 15
2.2	The Hardy–Leray inequality — 15
2.3	Optimality and nonattainability of the constant $\Lambda_{N,2}$ — 16
2.3.1	Some preliminaries on symmetrization — 17
2.3.2	Constant $\Lambda_{N,2}$ is the best constant and is not attained — 19
2.4	Hardy’s inequality in $W^{1,p}(\mathbb{R}^N)$, $1 < p < N$ — 21
2.5	Hardy–Leray inequality with remainder term — 23
2.5.1	A functional consequence — 28
3	Calderón–Zygmund theory and the Hardy–Leray potential — 29
3.1	Introduction — 29
3.1.1	Unbounded solution for $m > \frac{N}{2}$ — 30
3.1.2	Nonexistence result for $m = 1$ — 30
3.2	Summability of finite energy solutions — 33
3.2.1	Optimality of the condition on λ — 34
3.3	Existence of infinite energy solutions — 35
3.3.1	Complete blow-up for data in L^1 — 38
3.4	Necessary and sufficient conditions for solvability in weighted L^1 -spaces — 39
3.4.1	Uniqueness — 41
3.5	Further results — 41
3.5.1	A different way to obtain the critical value for λ — 41
3.5.2	A remark in the case $\lambda = \Lambda_{N,2}$ — 43
4	Effect of the Hardy–Leray potential in the solvability of semilinear elliptic equations — 45
4.1	Introduction — 45
4.2	The optimal power — 45

4.3	Some previous results for linear elliptic equations —	47
4.3.1	Elliptic equations with measure data —	47
4.3.2	Newtonian capacity —	48
4.3.3	The Ambrosetti–Rabinowitz mountain pass theorem —	54
4.4	Results on existence —	62
4.5	Results on nonexistence —	66
4.5.1	Complete blow-up —	68
4.6	Further results and comments —	69
5	The Hardy–Leray potential in semilinear heat equations —	71
5.1	Introduction —	71
5.2	Preliminaries and tools —	72
5.2.1	Local behavior of supersolutions of the linear equation —	74
5.2.2	A technical remark on existence —	77
5.3	Nonexistence results: $p \geq p_+(\lambda)$ —	79
5.4	Instantaneous and complete blow-up results —	83
5.4.1	Blow-up for the approximated problems when $p \geq p_+(\lambda)$ —	83
5.4.2	Blow-up when $p_n \rightarrow p_+(\lambda)$ —	84
5.5	Existence of solutions: $p < p_+(\lambda)$ —	87
5.6	Cauchy problem —	89
5.6.1	Subsolution blow-up in a finite time for small p —	90
5.6.2	Global supersolutions for $F(\lambda) < p < p_+(\lambda)$ —	92
5.6.3	Local existence results for $1 < p < p_+(\lambda)$ —	93
5.6.4	Global existence for $F(\lambda) < p < p_+(\lambda)$ and small data —	94
5.6.5	Blow-up result for $p < F(\lambda)$ —	94
5.7	Further results and remarks —	96
5.7.1	Problems involving the p -Laplacian heat equation —	96
5.7.2	Problems associated to the Caffarelli–Kohn–Nirenberg inequalities —	96
5.7.3	The borderline case $p = F(\lambda)$ —	97
6	Elliptic equations with a nonlinearity on the gradient and the Hardy–Leray potential —	103
6.1	Introduction —	103
6.2	Some auxiliary results —	104
6.2.1	Comparison results —	104
6.2.2	A quantitative version of the maximum principle —	112
6.2.3	Caffarelli–Kohn–Nirenberg inequalities —	113
6.3	Nonexistence results: exponent $q \geq q_+(\lambda)$ —	114
6.4	Blow-up result —	121
6.5	Existence result: $1 < p < q_+(\lambda)$ and $\lambda < \Lambda_{N,2}$ —	126
6.6	The critical case, $\lambda \equiv \Lambda_{N,2}$ and $p < \frac{N+2}{N}$ —	132

6.7	Further remarks — 135
7	The heat equation with nonlinearity on the gradient and the Hardy–Leray potential — 137
7.1	Introduction — 137
7.2	Preliminaries and tools — 139
7.2.1	Maximum principle and comparison results — 140
7.2.2	Local behavior of a very weak supersolution to problem (7.1.1) — 145
7.2.3	Passing to the limits in locally truncated problems — 147
7.3	Nonexistence and blow-up results — 154
7.3.1	Nonexistence — 154
7.3.2	Complete and instantaneous blow-up — 166
7.4	Existence results — 168
7.5	Cauchy problem — 173
7.5.1	A class of subsolutions to (7.5.2) for small p , blow-up in a finite time — 174
7.5.2	A class of global supersolutions to (7.5.2) for $F(\lambda) < p < q_+(\lambda)$ — 176
7.5.3	Local existence for $1 < p < q_+(\lambda)$ — 177
7.5.4	Global existence for small data and $F(\lambda) < p < q_+(\lambda)$ — 181
7.5.5	Blow-up in a finite time for $p < F(\lambda)$ and any positive initial datum — 184
7.6	Further remarks — 186
8	Fractional Laplacian type operators — 189
8.1	Introduction — 189
8.1.1	Riesz potentials — 192
8.1.2	Analytic continuation and the fractional Laplacian formula — 194
8.1.3	Some elementary properties of the fractional Laplacian — 197
8.2	Analytical preliminaries related to $(-\Delta)^s$ — 199
8.2.1	Some remarks on the regularity of solutions in the whole \mathbb{R}^N — 205
8.3	The Dirichlet problem: the variational framework — 211
8.3.1	Elementary estimates — 214
8.3.2	Elliptic Kato inequality — 217
8.3.3	Weak maximum principle and comparison results — 218
8.3.4	Some interpolation results — 220
8.4	Elliptic problem: finite energy setting — 221
8.4.1	Bounded solutions: Moser and Stampacchia methods — 222
8.4.2	The limit case $m = \frac{N}{2s}$: exponential summability — 225
8.4.3	A Calderón–Zygmund type result — 226
8.4.4	Further fractional regularity — 227
8.5	A fractional Picone inequality and applications to sublinear problems — 228

8.6	Nonvariational setting for elliptic problems: weak solutions —	232
8.6.1	Calderón–Zygmund type result for weak solutions —	236
8.7	Further results —	237
9	The fractional Hardy inequality —	239
9.1	Introduction —	239
9.2	The fractional Hardy inequality —	240
9.3	Normalizing the constants —	248
9.4	Local behavior of solutions of the elliptic equation —	250
9.5	<i>Ground state</i> representation —	252
9.6	Hardy’s inequality with remainder terms —	255
9.7	Further results and comments —	261
10	Calderón–Zygmund summability in the fractional setting —	263
10.1	Introduction and statement of the problem —	263
10.2	Functional setting: inequalities with weights —	265
10.2.1	Weighted Sobolev inequalities and applications —	269
10.2.2	Some compactness results —	277
10.2.3	Some numerical inequalities —	278
10.3	Weak Harnack inequality and local behavior of nonnegative supersolutions —	280
10.4	Optimal summability in the presence of Hardy potential —	291
10.4.1	Regularity of energy solutions —	291
10.4.2	About the optimality of the regularity results —	295
10.4.3	Nonvariational setting: weak solutions —	299
10.4.4	A necessary and sufficient condition for solvability —	301
10.5	Further results and comments —	303
11	Fractional semilinear elliptic problems —	305
11.1	Introduction —	305
11.2	Preliminaries —	306
11.2.1	A convergence tool by Brezis–Lieb —	309
11.2.2	Maximum principle for the fractional Laplacian —	310
11.2.3	The Pohozaev identity for the fractional Laplacian —	314
11.3	Existence of minimal solutions for $1 < p < p(\lambda, s)$ —	323
11.4	Existence of at least two nontrivial variational solutions if $1 < p \leq 2_s^* - 1$ —	329
11.4.1	Subcritical case —	329
11.4.2	The critical problem: $p = 2_s^* - 1$ —	342
11.5	Nonexistence for $p \geq p(\lambda, s)$: complete blow-up —	358
11.5.1	Complete blow-up —	362

11.6	Problems with the Hardy potential and nonlinear terms singular at the boundary — 365
11.7	Further comments — 372
11.7.1	Other operators — 372
12	The heat equation with fractional diffusion — 373
12.1	Introduction — 373
12.2	Finite energy setting — 373
12.3	Nonvariational setting: weak solutions — 376
12.4	<i>A priori</i> estimates and summability of the solutions — 379
12.4.1	Bounded solutions — 379
12.4.2	Summability of the solutions outside of the Aronson–Serrin zone — 384
12.4.3	Further summability results — 386
12.4.4	Parabolic Kato inequality — 388
12.5	The Cauchy problem for the fractional heat equation — 389
12.5.1	Some remarks on regularity of the solution to the Cauchy problem — 398
12.6	Regularity in bounded domains: relation between weak solutions and viscosity solutions — 404
12.7	The uniqueness result of Widder type — 406
12.7.1	Uniqueness for weak solutions — 411
12.7.2	Uniqueness for strong positive solutions — 419
12.7.3	About viscosity solutions — 424
12.8	Further results — 425
12.8.1	Some remarks about the fractional Widder theorem — 425
12.8.2	The Fujita exponent for the fractional heat equation — 428
13	The influence of the Hardy potential on the linear and semilinear fractional heat equations — 433
13.1	Introduction — 433
13.2	Functional framework: some preliminary results — 435
13.3	Weak Harnack inequality for a weighted problem — 441
13.4	The linear problem: dependence on the spectral parameter λ — 456
13.5	Existence and nonexistence results for a semilinear problem — 465
13.5.1	Nonexistence results for $p > p_+(\lambda, s)$, instantaneous and complete blow-up — 468
13.5.2	Existence results for $1 < p < p_+(\lambda, s)$ — 471
13.6	Further results and comments — 473
13.6.1	Fujita exponent depending on λ — 473
13.6.2	Other operators — 473

Bibliography — 475

Alphabetical Index — 489