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Field Theory

Introduction

1 Concept of a tensor

In theoretical physics all physical phenomena are described with the aid of various mathematical models in which the physical objects are associated with the mathematical ones. The mathematical objects describe the physical ones in space, specifying them in the different reference frames. Here, *each* point in space is associated with a set of coordinates, the number of them depending on the dimensionality of the space. The coordinates are usually denoted as x^i where the index i takes all possible values in accordance with the enumeration of the reference frame axes and is called *free index*. A set of coordinates pertaining to one point is referred to as *radius-vector* and is usually, in the three-dimensional case in particular, denoted with $\mathbf{r}$.

The properties of physical objects must not depend on the choice of reference frame and this determines the properties of mathematical objects corresponding to the physical ones. For example, the physical laws of conservation must be described with the aid of mathematical objects having the same form in various reference frames. These are called *invariants*. Various reference frames can be related with the aid of a certain coordinate transformation representing, from the formal mathematical viewpoint, a replacement written usually as $x^i(\mathbf{r}) = x'^i(\mathbf{r}')$. The prime is commonly ascribed to the radius-vector in the reference frame transformed with respect to the initial frame. Since for describing physical objects it is also necessary to apply the inverse transformations, the continuous and non-degenerate transformations alone should be used for replacing the coordinates for which $J = \det(\partial x^i / \partial x'^k) \neq 0$, J being the determinant of the Jacobi matrix, i. e. *Jacobian* of the given transformation.

The mathematical objects that remain unchangeable under *all* transformations (replacements) of the coordinates correspond to the invariants and these objects are referred to as *scalars*. A scalar has no free indices and thus can be denoted as φ. If in place of a simple scalar one specifies a *scalar field*, i. e. scalar $\varphi(\mathbf{r})$ as a function of the spatial point with coordinates $\mathbf{r}$, this function varies with the replacement of coordinates $\mathbf{r}(\mathbf{r}')$ so that the previous values would correspond to new coordinates of the previous points:

$$\varphi(\mathbf{r}(\mathbf{r}')) = \varphi'(\mathbf{r}').$$

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If the coordinate transformation is linear, the elements of the Jacobi matrix are simply numbers and the expression for the radius-vector of a point after transformation of the reference frame can be written in the following form:

$$x^i = \sum_k \frac{\partial x^i}{\partial x'^k} x'^k \equiv \frac{\partial x^i}{\partial x'^k} x'^k.$$

It is customary *always* to perform a summation over the doubly repeated so-called *dummy* index and thus the summation sign is usually omitted (rule of summation).

Provided that the coordinate transformation is nonlinear, the law of transformation is valid not for the radius-vector but for its differential

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k.$$

A *vector* or *contravariant vector* is called a set of quantities A^i varying under the transformation of a reference frame in the same way as the components of the radius-vector:

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k.$$

The derivative of the scalar field with respect to the radius-vector components, i.e. the gradient, is a multicomponent quantity as well. Under transformation of the reference frame its form will vary with the aid of the *inverse* transformation matrix. Such an object is referred to as a *covariant vector* (covector) and is written with a lower index:

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k} \frac{\partial x'^k}{\partial x^i}, \qquad u_i = \frac{\partial x'^k}{\partial x^i} u'_k.$$

For orthogonal transformations, the transformation laws for vectors and covectors coincide. In this case there is no necessity to distinguish between these two objects and use the upper and lower indices.

If the scalar field is doubly differentiated with respect to the radius-vector components, there appears a set of quantities described by two free indices transforming in the same way as the replacement of coordinates, namely as a product of components of two covariant vectors:

$$T_{ik} = \frac{\partial^2 \varphi}{\partial x^i \partial x^k} = \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^k} \frac{\partial^2 \varphi}{\partial x'^l \partial x'^m} \equiv \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^k} T'_{lm}.$$

Such mathematical objects are called *covariant tensors of second rank*. Accordingly, a (contravariant) tensor of second rank is a set of quantities with two upper indices which are transformed as a product of the corresponding components of two vectors. The rank of a tensor is determined by the number of its indices. In general, a tensor can be of *any* rank and have both upper and lower indices. The position of each index is important:

$$T^{ij\ldots}{}_{kl\ldots} = T'^{pq\ldots}{}_{rs\ldots} \frac{\partial x^i}{\partial x'^p} \frac{\partial x^j}{\partial x'^q} \cdots \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^s}{\partial x^l} \cdots.$$

In particular, a *scalar* is a tensor without indices, a *vector* is a tensor with one single upper index, and a *covariant vector* is a tensor with one single lower index.

The following operations are defined for tensors.

Contraction is a summation over a pair of recurring indices, one of them being the upper and the other being the lower, $T^{ij\ldots}{}_{il\ldots}$. Contracting over a single pair of indices decreases the rank of a tensor by two. The contraction is also defined for two tensors and here the number of pairs of same indices can be arbitrary, e.g. $T^{ij\cdots}{}_{kl\ldots}S^{kq\cdots}{}_{mj\ldots}$. Note that, in general,

$$T^{ij\ldots}{}_{kl\ldots}S^{kq\ldots}{}_{mj\ldots} \neq T^{ij\ldots}{}_{kl\ldots}S^{qk\ldots}{}_{jm\ldots}.$$

The contraction of two tensors of first rank is nothing but their scalar product $A^iB_i = (\mathbf{AB})$.

The *tensor* or *Kronecker product* is an elementwise multiplication of tensor components with sets of different indices $T^{ij\cdots}{}_{kl\ldots}S^{pq\cdots}{}_{mn\ldots}$. As a result, one obtains a tensor whose rank equals the sum of the ranks of the tensors in the product.

The *tensor equality* implies that two tensors with the same set of the lower and upper indices are equal

$$T^{ij\ldots}{}_{kl\ldots} = S^{ij\ldots}{}_{kl\ldots}$$

if the difference of the corresponding components of these tensors vanishes in an arbitrary reference frame. Thus, the equality of two tensors in N-dimensional space corresponds to a system of N^s equations, s being the tensor rank. It follows that vectors and tensors allow one to write physical relations in a form independent of the chosen reference frames. This is because both sides of the tensor equality transform according to the same rule under replacement of coordinates.

A combination of correctly constructed tensors results in another valid tensor provided the following rules of the index balance are observed:

- Each summand and each term of the equality must have the same sets of free indices, namely, indices with the same attributes must be in the identical (lower or upper) positions and the attribute of each free index runs once in each term.
- In each summand and in every part of the equality there may also be (or not) dummy indices, namely, every dummy index in each term either is absent or can be found exactly twice (once as an upper and once as a lower index).
- If any index occurs three times or more in one term, there is an error in the formula.
- If some index is found twice in the upper position or in the lower position, the formula has an error. (Otherwise, we deal with tensors corresponding to orthogonal transformations.)
- We can rename any free index if we change its name equally in all summands. Here, one should keep in mind that the new index name must differ from the names of the other indices used in every summand. Analogously, we can replace the free index with its numerical value.
- We can rename any dummy index in some summand, replacing simultaneously the name of both its positions. In this operation the new name of the index should not coincide with the names of other indices used in the summand.

One can obtain a tensor of second rank by differentiating the components of the radius-vector with respect to its proper components. This results in a symmetric tensor whose components are invariant with respect to any coordinate transformations:

$$\delta^i_k = \frac{\partial x^i}{\partial x^k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k \end{cases}$$

The tensor δ^i_k is usually referred to as the *Kronecker symbol*.

Among various invariants an especially important one is the *element of length*, its square being $(dl)^2 = dx^i dx_i$. Since this is a scalar, its value must remain unchanged by a replacement of coordinates:

$$(dl)^2 = dx^i\, dx_i = \frac{\partial x^i}{\partial x'^k}\frac{\partial x'^l}{\partial x^i} dx'^k dx'_l = g^l_k dx'^k dx'_l.$$

According to the definition, the factor in front of $dx'^k dx'_l$ is a mixed second-rank tensor which can be denoted as g^l_k. If both indices are lower ones, a covariant tensor of second rank can be introduced as

$$g_{kl} = \sum_i \frac{\partial x^i}{\partial x'^k}\frac{\partial x^i}{\partial x'^l},$$

and thus the element of length can be written using the contravariant vector components alone:

$$(dl)^2 = g_{ik} dx^i\, dx^k.$$

The tensor g_{ik} is called the *metric*. It is easy to see that the contravariant tensor g^{ik} proves to be inverse to g_{ik}, i. e. $g_{ik}g^{kl} = \delta^l_i$. With replacement of variables the metric tensor transforms according to the definition

$$g_{ik} = \frac{\partial x'^j}{\partial x^i}\frac{\partial x'^l}{\partial x^k} g'_{jl}$$

For a second-rank tensor, one can calculate its determinant. Calculating the determinant for both sides we arrive at $g = (J^{-1})^2 g'$ or $\sqrt{g'} = |J|\sqrt{g}$.

The metric tensor can be used for raising or lowering the indices:

$$A_i = g_{ik}A^k, \quad F^i{}_j = g^{ik}F_{kj}, \quad \textit{also}$$
$$A^iB_i = g_{ik}A^iB^k = g^{ik}A_kB_i.$$

There exists one more useful invariant tensor which has the same rank as the space in which a physical object is treated. This is a completely antisymmetric tensor whose components change sign under permutation of any pair of indices and whose nonzero components are ± 1. For definiteness, we consider the three-dimensional case. Then, the completely antisymmetric tensor is a third-rank tensor, $e^{ikl} = -e^{kil} = e^{kli}$. Due to the antisymmetric property all six components differ only in sign. It is agreed to put +1 for the component having the right ordering of indices, i. e. in Cartesian coordinates $e^{123} = e^{xyz} = +1$. Let us write the relation of the tensor components obtained after a coordinate transformation with the initial one:

$$e^{ikl} = \frac{\partial x^i}{\partial x'^p}\frac{\partial x^k}{\partial x'^q}\frac{\partial x^l}{\partial x'^r} e'^{pqr}.$$

A summation over twice recurring indices taking account of the sign interchange gives

$$e^{ikl} = Je'^{ikl}$$

where J is the Jacobian of the transformation. If the Jacobian of the transformation equals unity, e.g. in the case of pure rotations, the tensor components remain unchanged[1] . In the three-dimensional case the tensor e^{ikl} is usually called the *Levi-Civita symbol*.

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1 The availability of the Jacobian indicates that a tensor defined in such manner has the properties of a density.

2 Vectors and tensors in Euclidean space

In Euclidean space there exist a preferred systems of coordinates called *Cartesian coordinates* in which the components of the metric tensor g_{ij} $(i, j = 1, 2, 3)$ are given by the unit matrix. In this case the raising or lowering of indeces does not change the values of tensor components. The Jacobi matrices for coordinate transformations are always orthogonal, $J^{-1} = J^T$, and vectors with upper and lower indices transform identically.

With the exception of specified cases we will use Cartesian coordinates for describing Euclidean space: $\mathbf{r} = (x, y, z) = (x_1, x_2, x_3)$, *allowing us to disregard the difference in the upper and lower indices.* We will use Latin characters for denoting tensor indices in Euclidean space and write the indices as subscripts.

Usually, in Euclidean space a vector is denoted either by a bold character or by an arrow over its symbol.

Transformations of the Cartesian reference frame for which the reference origin remains invariant reduce to rotations around some axes, reflections in planes, and inversion. For rotations, the Jacobian is $J = +1$, and for reflections and inversion one has $J = -1$. Therefore, the tensor e_{ikl} behaves as a tensor for rotational transformations but its components change sign for reflections. In other words, this tensor has an attribute different from that of a genuine tensor. Thus, the tensor e_{ikl} is often referred to as a *pseudotensor*.

The scalar, cross (vector) and scalar triple (mixed) products in tensor notation take the form

$$(\mathbf{a}\ \mathbf{b}) = a_i b_i = \delta_{ij} a_i b_j, \quad [\mathbf{a} \times \mathbf{b}]_i = e_{ijk} a_j b_k, \quad (\mathbf{a}\ [\mathbf{b} \times \mathbf{c}]) = e_{ijk} a_i b_j c_k.$$

As one sees, the cross product is a contraction of a pseudotensor and two vectors over two pairs of indices, and thus the resultant vector is not genuine and is usually referred to as *pseudovector* or *axial vector*.

The differential operator or *del*, presented by the *nabla symbol* $\boldsymbol{\nabla}$, is in Cartesian coordinates a vector with the following components:

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \quad \nabla_i = \frac{\partial}{\partial x_i}.$$

In three-dimensional Euclidean space the basic functions, having invariant meaning and thus often used, read in tensor notation as follows:

$$\boldsymbol{\nabla}\varphi = \operatorname{grad}\varphi \quad \text{or} \quad \operatorname{grad}\varphi|_i = \nabla_i \varphi = \partial\varphi/\partial x_i,$$

$$(\boldsymbol{\nabla}\ \mathbf{a}) = \operatorname{div}\mathbf{a} = \nabla_i a_i = \partial a_i/\partial x_i,$$

$$[\boldsymbol{\nabla} \times \mathbf{a}] = \operatorname{curl}\mathbf{a} \quad \text{or} \quad \operatorname{curl}\mathbf{a}|_i = e_{ijk}\nabla_j a_k = e_{ijk}\partial a_k/\partial x_j,$$

$$(\mathbf{a}\ \boldsymbol{\nabla}) = \nabla_{\mathbf{a}} = a_i \nabla_i \text{ (differential operator in the } \mathbf{a} \text{ direction)},$$

$$(\boldsymbol{\nabla}\ \boldsymbol{\nabla}) = \operatorname{div}\operatorname{grad} = \nabla^2 = \nabla_i \nabla_i = \Delta \text{ (Laplace operator)},$$

$$\operatorname{div}\mathbf{r} = \nabla_i x_i = 3,$$

$$\operatorname{curl}\mathbf{r}|_i = [\boldsymbol{\nabla} \times \mathbf{r}]_i = e_{ijk}\nabla_j x_k = 0,$$

$$a_i \nabla_i x_j = (\mathbf{a}\ \boldsymbol{\nabla})\mathbf{r}|_j = a_j \quad \text{or} \quad (\mathbf{a}\ \boldsymbol{\nabla})\mathbf{r} = \mathbf{a},$$

$$\operatorname{grad} r = \mathbf{r}/r = \mathbf{n}.$$

The integral Gauss theorem in invariant form is given by

$$\iiint_V \operatorname{div} \mathbf{A}\, dV \equiv \iiint_V (\boldsymbol{\nabla}\, \mathbf{A})\, dV = \oint_S (\mathbf{A}\, d\mathbf{S}) = \oint_S (\mathbf{A}\, \mathbf{n})\, dS.$$

The surface element or the normal is implied to be directed *outside*.

The Gauss theorem in tensor form reads as

$$\iiint_V \operatorname{div} \mathbf{A}\, dV \equiv \iiint_V \frac{\partial A_i}{\partial x_i}\, dV = \oint_S A_i\, dS_i \equiv \oint_S A_i n_i\, dS.$$

The integral Stokes theorem in invariant form is given by

$$\iint_S (\operatorname{curl}\ \mathbf{A}\, d\mathbf{S}) \equiv \iint_S (d\mathbf{S}\ [\boldsymbol{\nabla} \times \mathbf{A}]) = \oint_L (\mathbf{A}\, d\mathbf{l}).$$

The Stokes theorem in tensor form reads as

$$\iint_S (\operatorname{curl}\ \mathbf{A}\, d\mathbf{S}) \equiv \iint_S e_{ijk} \frac{\partial A_k}{\partial x_j}\, dS_i = \oint_L A_i\, dx_i.$$

3 Vectors and tensors in Minkowski space

In the four-dimensional Minkowski space there also exists a preferred system of coordinates (*Lorentz coordinates*) usually referred to as *reference frames*. In Lorentz coordinates the components of the metric tensor (*Minkowski metric*) are written as $g_{\mu\nu} = \operatorname{diag}(+1, -1, -1, -1)$ $(\mu, \nu = 0, 1, 2, 3)$. In this case, raising or lowering an index may change the sign of the tensor component, resulting in the necessity to distinguish between upper and lower indices.

With the aid of the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ one can raise or lower the indices in the following way:

$$g_{\mu\nu} = g^{\mu\nu} = \operatorname{diag}(+1, -1, -1, -1) = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.1}$$

$$v_\mu = v^\nu g_{\mu\nu}, \quad v^\mu = v_\nu g^{\mu\nu}, \quad T^{\mu\nu\lambda} = T^{\mu}{}_{\varkappa}{}^{\lambda} g^{\varkappa\nu}, \quad etc.$$

Greek characters will stand for the tensor indices in Minkowski space.

With the exception of a few specified cases we use Lorentz coordinates for describing Minkowski space, $x^\mu \equiv \underline{x} = (ct,\ x,\ y,\ z) = (ct, \mathbf{r}) = (x^0, x^1, x^2, x^3)$, c being the speed of light. A point in Minkowski space is called an *event*. It is obvious that

$$x_\mu = (x_0, x_1, x_2, x_3) = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{r}).$$

A vector in Minkowski space is often denoted by an underlined character, $A^\mu \equiv \underline{A}$.

The four-dimensional distance between two events s, the length of a four-dimensional radius-vector (or, for short, a 4-radius-vector), is called an *interval*, its square being equal to

$$s^2 = g_{\mu\nu}(x^\mu x^\nu) \equiv \underline{x}\underline{x} = (ct)^2 - \mathbf{r}^2.$$

In general the square s^2 of an interval may be positive, negative, or zero. The case $s^2 > 0$ is referred to as a timelike one (two events may occur at the same point in space but at different time points). The case $s^2 < 0$, this is said to be spacelike. For the lightlike line one finds $s = 0$, this represents the light or zero interval.

The completely antisymmetric tensor $e^{\mu\nu\alpha\beta}$ is a tensor with respect to the transformations with unit Jacobian, i. e. for rotations and other transformations conserving the volume and orientation of the basis. In the four-dimensional case it is usually defined by

$$e^{\varkappa\lambda\mu\nu} = -e^{\lambda\varkappa\mu\nu} = -e^{\varkappa\mu\lambda\nu} = -e^{\varkappa\lambda\nu\mu} = -e^{\nu\lambda\mu\varkappa}, \quad e^{0123} = +1.$$

With the aid of the completely antisymmetric tensor $e_{\varkappa\lambda\mu\nu}$ in Minkowski space one can introduce the objects dual to scalar, vector, and antisymmetric tensors of rank 2, 3 and 4:

$$\tilde{\varphi}^{\varkappa\lambda\mu\nu} = \frac{1}{0!}e^{\varkappa\lambda\mu\nu}\varphi, \quad \tilde{A}^{\lambda\mu\nu} = \frac{1}{1!}e^{\varkappa\lambda\mu\nu}A_{\varkappa}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2!}e^{\varkappa\lambda\mu\nu}F_{\varkappa\lambda},$$
$$\tilde{G}^{\nu} = \frac{1}{3!}e^{\varkappa\lambda\mu\nu}G_{\varkappa\lambda\mu}, \quad \tilde{H} = \frac{1}{4!}e^{\varkappa\lambda\mu\nu}H_{\varkappa\lambda\mu\nu}.$$

The differential operator (*4-gradient operator*) in linear coordinates represents a covariant vector and reads as

$$\frac{\partial}{\partial x^{\mu}} = \partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \ \boldsymbol{\nabla}\right).$$

With the exception of specified cases we consider in the following only linear coordinate transformations conserving the Minkowski metric. A set of such transformations embraces translations, spatial reflections, reversion of time, spatial rotations, Lorentz transformations, and their possible combinations.

4 Relativistic kinematics

The Lorentz transformation determines the transfer from one inertial reference frame to another. The rotation-free Lorentz transformation is also called a *boost*. Contravariant vectors and tensors in four-dimensional space are simply referred to as *four-vectors* and *four-tensors*. The standard Lorentz transformation of an arbitrary 4-vector A^{μ}, determined for the case when the velocity V of the reference frame moving relative to the laboratory frame is directed along the x axis, has the form[2]

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix}, \quad \beta = \frac{V}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Covariant vectors transform with the aid of the transposed inverse matrix which can be obtained by replacing V with $-V$. The boosts in the y or z direction can be derived with the corresponding permutations of rows and columns in the transformation matrix.

A contravariant vector and its components are usually written as $A^{\mu} \equiv (A^0, \mathbf{A}) \equiv (A^0, A^i)$, and a covariant vector reads correspondingly as $A_{\mu} \equiv (A^0, -\mathbf{A}) \equiv (A^0, -A^i) = (A_0, A_i)$.

The scalar or dot product of two vectors equals

$$A^{\mu}B_{\mu} = A_{\mu}B^{\mu} = A^{\mu}B^{\nu}g_{\mu\nu} = A_{\mu}B_{\nu}g^{\mu\nu} = A^0B^0 - \mathbf{AB} = A^0B^0 - A^iB^i.$$

2 The *rapidity* θ, a real parameter related to the velocity as $\tanh\theta = V/c$, can also be useful for describing the Lorentz transformation.

The interval between events, associated with the particle motion, is expressed in terms of the particle velocity v as the invariant $ds = cd\tau = cdt\sqrt{1-(v/c)^2}$. Here, τ is *the proper time* of a particle, i.e. the time in the frame in which the particle is at rest. The proper time τ is often used for the definition of various kinematic quantities, differing from the interval along the timelike world line by a factor of light speed c.

The 4-velocity and 4-acceleration have the forms

$$u^\mu = \frac{dx^\mu}{ds} = \left(\gamma, \gamma\frac{\mathbf{v}}{c}\right), \qquad w^\mu = \frac{du^\mu}{ds} = \frac{d^2x^\mu}{ds^2}. \tag{1.2}$$

From the definition of 4-velocity and 4-acceleration we have

$$u^\mu u_\mu = \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = \frac{dx^\mu\, dx_\mu}{ds\, ds} = \frac{ds^2}{ds^2} = 1, \qquad u^\mu w_\mu = \frac{1}{2}\frac{d}{ds}u^\mu u_\mu = 0.$$

The 4-momentum of a particle with mass m reads as

$$p^\mu = mcu^\mu = \left(\frac{\mathcal{E}}{c},\ \mathbf{p}\right) = (m\gamma c,\ m\gamma\mathbf{v}), \quad \mathbf{p} = \frac{\mathcal{E}\mathbf{v}}{c^2} \tag{1.3}$$

where $\mathcal{E}$ is the energy and $\mathbf{p}$ is the three-dimensional momentum.

The rest energy $\mathcal{E}_0 = mc^2$ is unambiguously associated with the mass of a particle[3] and is also an invariant.

The general definition of mass can be derived from the invariant

$$p^\mu p_\mu = (\mathcal{E}/c)^2 - \mathbf{p}^{\,2} = m^2c^2.$$

For a massless particle, the 4-velocity is not a well-defined quantity but the 4-momentum can readily be determined, obeying the general relation $p^\mu p_\mu = m^2c^2 = 0$ in this case.

For a system of particles, the 4-momentum conservation law holds:

$$\sum_a p_a^\mu = \sum_b p'^\mu_b \tag{1.4}$$

where the sum is taken over all particles before and after any interaction among them, e.g. scattering, decay, reactions, etc. The total energy and spatial momentum are conserved. However, both the number and the type of particles can vary as well as the total mass of all particles. Thus, on the left-hand and right-hand sides of Eq. (1.4) the index of summation is denoted by different characters. The 4-momentum conservation law, written in the form (1.4), is very convenient for solving various problems with the aid of four-dimensional invariants obtained by separating the squared 4-momentum of one or several particles from the above-mentioned equation. For example, one can write for two particles of known masses

$$p'^\mu_2 p'_{2\mu} = (m_2c)^2 = (p_1^\mu + p_2^\mu - p'^\mu_1)(p_{1\mu} + p_{2\mu} - p'_{1\mu}).$$

Application of the method of 4-invariants depends on the process under consideration.

An *elastic process* is a process in which the amount of particles and their types remain unchanged and, in particular, the particle masses do not vary either.

3 Commonly, in the theories of relativity and particle physics a system of units is used in which $c = 1$. In this case, the electron volt (eV) is used as a common unit of measurement for energy, momentum and mass, with the following multiple prefixes: 1keV= 10^3eV, 1 MeV= 10^6eV, 1 GeV= 10^9eV, 1 TeV= 10^{12}eV. The usage of such units is frequently not specified.

An *inelastic process* is a process with varying numbers and/or types and, in particular, masses of particles.

Among kinematic problems associated with inelastic processes a very important problem is the determination of the threshold of reactions in the production of particles that differ from the initial ones. The *energy threshold* of the reaction is the minimum kinetic energy $T = \mathcal{E} - mc^2$ of the incident particle, necessary for the creation of a new particle. For the system of particles, one can determine the effective mass as an energy in the center-of-inertia frame, using the invariance of the squared 4-momentum:

$$M_{\text{eff}}^2 c^2 = p_{\text{coll}}^{\mu} p_{\text{coll}\ \mu}. \tag{1.5}$$

As follows from Eq. (1.5), the energy will be minimal if the particles are at rest in the center-of-inertia frame. Then, the effective mass equals the sum of the masses of all particles after the reaction:

$$M_{\text{eff}} = \sum_b m_b.$$

5 The Maxwell equations

(1) The Maxwell equations in three-dimensional form. The Maxwell equations are commonly written in the form of two pairs, one without sources and the other with sources. In three-dimensional (in differential) form the equations read as

$$\begin{cases} \operatorname{div} \mathbf{H} = 0, \\ \operatorname{curl} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, \end{cases} \qquad \text{(the first pair)}$$

$$\begin{cases} \operatorname{div} \mathbf{E} = 4\pi\rho, \\ \operatorname{curl} \mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{4\pi}{c}\mathbf{j}. \end{cases} \qquad \text{(the second pair)}$$

The *continuity equation* or charge conservation law results from the second pair of the Maxwell equations:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\ \mathbf{j} = 0.$$

The densities of charge and current induced by a point particle are equal to

$$\rho(\mathbf{r}, t) = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}_a(t)), \quad \mathbf{j}(\mathbf{r}, t) = \sum_a e_a \mathbf{v}_a(t) \delta(\mathbf{r} - \mathbf{r}_a(t)). \tag{1.6}$$

The first pair of the Maxwell equations allows us to parameterize the electromagnetic field by introducing the scalar φ and vector $\mathbf{A}$ potentials:

$$\mathbf{H} = \operatorname{curl}\ \mathbf{A}, \quad \mathbf{E} = -\operatorname{grad}\ \varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}.$$

(2) The Maxwell equations in four-dimensional form. The scalar and vector potentials treated together form the components of a 4-vector $A^\mu = (\varphi, \mathbf{A})$. The electric and magnetic fields can be expressed via the components of the electromagnetic field 4-tensor. The latter is determined with the aid of derivatives of the 4-potential components with respect to the components of the 4-radius-vector according to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The components of electric **E** and magnetic **H** fields are connected with the components of the antisymmetric field 4-tensor $F_{\mu\nu}$ as follows:

$$E_k = F_{0k}\ , \quad H_k = -\frac{1}{2}\, e_{kij} F^{ij} .$$

This can clearly be represented in the matrix form

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix} .$$

Introducing the electromagnetic field tensor allows one to write the Maxwell equations in four-dimensional form. Then, the first pair of the equations is determined by the 4-divergence of the dual tensor $\tilde{F}^{\mu\nu} = 1\ /2\ e^{\mu\nu\alpha\beta} F_{\alpha\beta}$:

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \qquad \text{(the first pair),} \tag{1.7}$$

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \quad \text{(the second pair).} \tag{1.8}$$

The 4-vector of the electric current density reads $j^\mu = (c\rho, \mathbf{j})$ and the continuity equation is the divergence of the 4-current density:

$$\frac{\partial}{\partial x^\mu} j^\mu = \partial_\mu j^\mu = 0.$$

The potentials of the given electromagnetic field are ambiguously determined. In fact, due to *gauge symmetry* it is always possible to perform a *gauge transformation* of the form

$$\mathbf{A}' = \mathbf{A} - \text{grad}\ f, \quad \varphi' = \varphi + \frac{1}{c}\frac{\partial f}{\partial t} \quad \text{or} \quad A'_\mu = A_\mu + \partial_\mu f$$

where $f(t, x, y, z)$ is an arbitrary function. The transformations considered leave the strengths of electromagnetic field **E** and **H** invariant.

In order to restrict in part the randomness associated with gauge symmetry, additional *gauge conditions*, simply called the *gauge*, are imposed on the potentials. The following gauge conditions are most common:

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial \varphi}{\partial t} + \text{div}\ \mathbf{A} = 0 \qquad \text{(the Lorentz gauge),}$$

$$\nabla \mathbf{A} = 0 \qquad \text{(the Coulomb gauge),}$$

$$\varphi = 0 \qquad \text{(the Weyl gauge).}$$

Substituting the explicit expression for tensor $F^{\mu\nu}$ given in the 4-potential components into the second pair of Eqs. (1.8), we have

$$\partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu .$$

For this case, it is convenient to choose the Lorentz gauge condition in order to obtain the wave equation for the 4-potential components (*the d'Alembert equation*):

$$\Box A^{\mu} = -\frac{4\pi}{c} j^{\mu}, \quad \text{where} - \partial_{\nu}\partial^{\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box. \tag{1.9}$$

Here, the box $\Box$ stands for the *d'Alembertian* or *d'Alembert operator* which can be treated as the Laplace operator in Minkowski space.

After choosing the Lorentz gauge, it is still possible to subject the potentials to residual gauge transformations which also do not modify the fields **E** and **H** and do not disturb the Lorentz gauge:

$$A'_{\mu} = A_{\mu} + \partial_{\mu} f, \quad \text{and} \quad \Box f = 0.$$

The electric and magnetic fields, as 4-tensor components, transform with the transition from one reference frame to another according to the Lorentz transformations for the components of a 4-tensor. In the case of the usual Lorentz transformation we have

$$H_x = H'_x, \quad H_y = \frac{H'_y - \beta E'_z}{\sqrt{1-\beta^2}}, \quad H_z = \frac{H'_z + \beta E'_y}{\sqrt{1-\beta^2}},$$
$$E_x = E'_x, \quad E_y = \frac{E'_y + \beta H'_z}{\sqrt{1-\beta^2}}, \quad E_z = \frac{E'_z - \beta H'_y}{\sqrt{1-\beta^2}}.$$

Note that it is simpler to derive the rule for the transformation of the electric field components from the transform of the dual tensor $\tilde{F}^{\mu\nu}$.

The electromagnetic field tensor $F^{\mu\nu}$ together with the dual tensor $\tilde{F}^{\mu\nu}$ allows one to determine two invariants of the electromagnetic field:

$$F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{H}^2 - \mathbf{E}^2) = I_1 = inv; \quad F^{\mu\nu}\tilde{F}_{\mu\nu} = 4(\mathbf{HE}) = I_2 = inv. \tag{1.10}$$

(3) The action function for the electromagnetic field. The equations for the electromagnetic field and for the motion of charged particles in the general form can be obtained starting from *the principle of least action*. In its most general form the action function is represented as a sum of terms describing the separate subsystems of the system in question and their interaction. Hence, the action for the electromagnetic field and electrically charged particles should consist of these three contributions:

$$S = S_{\text{part}} + S_{\text{field}} + S_{\text{int}}, \tag{1.11}$$

i.e. actions for particles, field and particle-field interactions. On account of the universal principle of least action, the action must be a scalar, i.e. a 4-invariant in our case. Deriving specific expressions for the action, one should keep in mind additional conditions. In particular, the theory obtained should be gauge-invariant and, in addition, provide a *principle of correspondence*, i.e. crossover to the known expressions in the nonrelativistic limit.

On derivation of the equations for a field interacting with charged particles it is necessary to employ the second and third terms in (1.11). It is convenient to write these terms in the form of the action for the electromagnetic field with sources,

$$S[A_{\mu}(\underline{x})] = \int \left(-\frac{F_{\mu\nu}F^{\mu\nu}}{16\pi c} - \frac{1}{c} j^{\mu} A_{\mu} \right) d^4x = \frac{1}{c} \int \mathcal{L} \, d^4x, \tag{1.12}$$

$\mathcal{L}$ being the Lagrangian density. The integrals along the world lines of particles are represented as integrals over space-time, and the current density 4-vector is introduced according to

$$j^{\mu}(\underline{x}) = \sum_a \int \delta^{(4)}(\underline{x} - \underline{\mathrm{x}}_a(s_a))\, e_a \frac{d\mathrm{x}_a^{\mu}}{ds_a}\, ds_a. \tag{1.13}$$

Here, $\mathrm{x}_a^{\mu}(s_a)$ is the 4-radius-vector of the ath particle, $d\mathrm{x}_a^{\mu}/ds_a$ is its 4-velocity (1.2), $s_a = c\tau_a$ is the interval associated with the proper time of a particle, and $\delta^{(4)}(\underline{x} - \underline{\mathrm{x}}_a(s_a)) = \delta(ct - \mathrm{x}_a^0(s_a))\,\delta\,(\mathbf{r} - \mathbf{r}_a(s_a))$ is the four-dimensional δ-function. Integrating in (1.13) over ds_a with the aid of Eq. (A.7), we arrive at the familiar formula (1.6). Note that the Lagrangian density $\mathcal{L}$ is a scalar function since the element of four-dimensional volume is a scalar as well.

The components of 4-potential of the field A_{μ} are here the independent generalized coordinates in which the variation of action is calculated. Under the condition of vanishing variation of the action with respect to A_{μ}, i.e. $\delta S[A_{\mu}(\underline{x})] = 0$, we obtain the Euler-Lagrange equation for the electromagnetic field:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu}} - \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\nu} A_{\mu})} \right) = 0.$$

The second pair of the Maxwell equations (1.8) results from this equation. According to the definition of tensor $F_{\mu\nu}$ the first pair of Eqs. (1.7) is satisfied identically.

(4) The energy-momentum tensor. Based on the action for the electromagnetic field S_{field}, see (1.11) or the first term in (1.12), one can find the so-called canonical energy-momentum tensor:

$$\tilde{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\alpha})} \partial^{\nu} A_{\alpha} - g^{\mu\nu}\mathcal{L}. \tag{1.14}$$

The definition of the energy-momentum tensor (1.14), in general, is not unambiguous. In fact, any tensor $\delta T^{\mu\nu}$ with zero 4-divergence can be added to the initial tensor, i.e.

$$T^{\mu\nu} = \tilde{T}^{\mu\nu} + \delta T^{\mu\nu}, \qquad \partial_{\mu}(\delta T^{\mu\nu}) = 0.$$

The latter is commonly chosen so that the final energy-momentum tensor would be symmetric, i.e. $T_{\mu\nu} = T_{\nu\mu}$.

The symmetric energy-momentum tensor for the electromagnetic field reads

$$T^{\mu\nu} = \frac{1}{4\pi} \left(\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F^{\nu}{}_{\alpha} \right).$$

In three-dimensional notation, the tensor splits into blocks:

$$T^{\mu\nu} = \begin{pmatrix} w & S_x/c & S_y/c & S_z/c \\ S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} w & \mathbf{S}/c \\ \mathbf{S}/c & \sigma_{ij} \end{pmatrix} \tag{1.15}$$

where the energy density w and the energy flux density vector or *Poynting vector* $\mathbf{S}$ for the electromagnetic field are equal to

$$w = \frac{1}{8\pi} \left(\mathbf{E}^2 + \mathbf{H}^2 \right), \quad \mathbf{S} = \frac{c}{4\pi} \left[\mathbf{E} \times \mathbf{H} \right].$$

The momentum density $\mathbf{g}$ and the *momentum flux density tensor*[4] σ_{ij} of electromagnetic field are defined as

$$\mathbf{g} = \frac{1}{4\pi c}[\mathbf{E} \times \mathbf{H}] = \frac{\mathbf{S}}{c^2}, \quad \sigma_{ij} = w\delta_{ij} - \frac{1}{4\pi}(E_i E_j + H_i H_j). \tag{1.16}$$

For a closed system consisting of an electromagnetic field and particles interacting with the field, the *energy-momentum conservation law* holds:

$$\partial_\nu T^{\mu\nu}_{\text{tot}} = 0. \tag{1.17}$$

This is an obvious illustration of the Noether theorem stating that the invariance of the Lagrangian with respect to any one-parameter transformation yields the corresponding local conserving current. In our case the energy-momentum conservation relates to the invariance of the Lagrangian with respect to the space-time translation $x^\mu \to x^\mu + a^\mu$.

If we write the expression for the 4-divergence of the energy-momentum tensor of the *field alone*, we obtain the following relation instead of (1.17):

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = -\frac{1}{c}F^{\mu\nu} j_\nu = -\mathcal{F}^\mu \tag{1.18}$$

where $\mathcal{F}^\mu$ is the density of the force which the field exerts on the charge system. In three-dimensional form the above equation reads as

$$\begin{cases} \dfrac{\partial w}{\partial t} + \mathrm{div}\mathbf{S} = -(\mathbf{jE}), \\ \dfrac{\partial \mathrm{g}_i}{\partial t} + \dfrac{\partial \sigma_{ij}}{\partial x_j} = -\rho E_i - \dfrac{1}{c}[\mathbf{j} \times \mathbf{H}]_i. \end{cases} \tag{1.19}$$

6 Motion of a charged particle in an external field

The equations of motion for a charged particle in an external electromagnetic field can be derived from the general principle of least action. In our case it is sufficient to use only the first and third terms in Eq. (1.11). The action for a *single* particle interacting with the field and satisfying all additional conditions can be written as

$$S[\underline{\mathrm{x}}(s)] = -\int_a^b \left(mc\sqrt{\frac{d\mathrm{x}^\mu}{ds}\frac{d\mathrm{x}_\mu}{ds}} + \frac{e}{c}A_\mu \frac{d\mathrm{x}^\mu}{ds} \right) ds = \int_a^b L ds \tag{1.20}$$

where the integration limits indicate the initial and final positions of a particle at the world line and L is the corresponding Lagrange function or *Lagrangian*. The variation of the action (1.20) with respect to the particle coordinates $\mathrm{x}^\mu(s)$ for a fixed field A_μ yields the usual Euler-Lagrange equation, resulting in the equation of motion of a particle in covariant form:

$$mc\frac{du^\mu}{ds} = \frac{e}{c}F^{\mu\nu}u_\nu. \tag{1.21}$$

4 The negative value of the momentum flux density tensor, i.e. $-\sigma_{ij}$, is called the *Maxwell stress tensor*.

Here, $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$, $u^\mu = dx^\mu/ds$ is the 4-velocity of a particle (1.2). In three-dimensional form these equations read as

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}\,[\mathbf{v}\times\mathbf{H}], \tag{1.22}$$

$$\frac{d\mathcal{E}}{dt} = e(\mathbf{E}\mathbf{v}). \tag{1.23}$$

The right-hand side of Eq. (1.22) represents the *Lorentz force*, and Eq. (1.23), time component of Eq. (1.21), is nothing but the temporal variation of the energy of a particle.

For systems of extended charges and currents with the corresponding densities ρ and $\mathbf{j}$, the notion of the force density $\boldsymbol{\mathcal{F}}$ is commonly introduced as (see (1.18))

$$\boldsymbol{\mathcal{F}} = \rho\mathbf{E} + \frac{1}{c}[\mathbf{j}\times\mathbf{H}] \quad \text{so that} \quad \mathbf{F} = \int \boldsymbol{\mathcal{F}} d^3\,\mathbf{r}.$$

Then, the time-law of energy in the system is given by

$$\frac{d\mathcal{E}}{dt} = \int d^3\,\mathbf{r}(\boldsymbol{\mathcal{F}}\mathbf{v}) = \int d^3\mathbf{r}\rho(\mathbf{v}\mathbf{E}) \equiv \int d^3\,\mathbf{r}(\mathbf{E}\mathbf{j}).$$

The law of energy conservation in a system of charged particles and an electromagnetic field can be represented in integral form as (compare Eq. (1.19))

$$\frac{d\mathcal{E}}{dt} = -\frac{dW}{dt} - \int d^3\,\mathbf{r}\ \mathrm{div}\mathbf{S} = -\frac{dW}{dt} - \oint (\mathbf{S}d\mathbf{f}).$$

Here,

$$W = \frac{1}{8\pi}\int d^3\,\mathbf{r}(\mathbf{E}^2 + \mathbf{H}^2)$$

is the electromagnetic field energy in the volume across whose boundary the field energy, transferred via the Poynting vector, flows in the outward direction, and $\mathcal{E}$ is the energy of the charged particles in the volume considered.

Calculating the generalized (or *canonical*) 4-momentum $P_\mu = -\partial L/\partial u^\mu = mcu_\mu + (e/c)A_\mu$ according to the general rules of Eq. (1.20), we find that the generalized momentum of a charged particle in the electromagnetic field differs from the ordinary kinetic momentum (1.3):

$$\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}.$$

The Hamiltonian function for a particle in the external field is equal to

$$\mathcal{H}(\mathbf{P},\mathbf{q}) = \sqrt{\mathbf{p}^2c^2 + m^2c^4} + e\varphi = c\sqrt{\left(\mathbf{P} - \frac{e}{c}\mathbf{A}(\mathbf{q})\right)^2 + m^2c^2} + e\varphi(\mathbf{q}).$$

7 Static electromagnetic field

The wave equations for the components of the 4-potential (the d'Alembert equation, 1.9) represent a set of linear nonuniform equations and their solution can be written as

$$A^\mu = A^\mu_f + A^\mu_e$$

where A_f^μ is the general solution of the uniform equation and A_e^μ is a partial solution of the nonuniform equation determined by the 4-current distribution. The general solution of the uniform equation represents the *free* electromagnetic field and a partial solution of the nonuniform equation can be obtained with the aid of the Green function satisfying the equation with the point source:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{1.24}$$

Hence, the solution for A_e^μ reads

$$A_e^\mu = \frac{1}{c}\int j^\mu(\mathbf{r}', t')G(\mathbf{r}, t; \mathbf{r}', t')d^3\mathbf{r}'dt'. \tag{1.25}$$

The physical conditions are satisfied by the solution of Eq. (1.24) in the form of the *retarded* Green function

$$G(\mathbf{r} - \mathbf{r}'; t - t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}\delta\left(t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right). \tag{1.26}$$

A partial solution for the components of the 4-potential (1.25) has, after integration in time, the form of the *retarded potentials*

$$A_e^\mu(\mathbf{r}, t) = \frac{1}{c}\int \frac{j^\mu(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \tag{1.27}$$

The partial solution of the wave equations has the simplest form in the case of a *static* distribution of charges and *stationary* currents in which case the integrand function is time-independent. In this case the solution can be treated regardless of the scalar or vector potentials, resulting in static electric and magnetic fields.

The scalar potential induced at point **R** equals

$$\varphi(\mathbf{R}) = \int \frac{\rho(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|}\, d^3\mathbf{r} = \sum_a \frac{e_a}{|\mathbf{R} - \mathbf{r}_a|}.$$

Here, the expressions are given for a system of extended and point charges. For distances far from the charge system $|\mathbf{R}| \gg |\mathbf{r}|$ or $|\mathbf{R}| \gg |\mathbf{r}_a|$, the expansion in *multipole moments* is valid

$$\varphi(\mathbf{R}) = \varphi^{(0)}(\mathbf{R}) + \varphi^{(1)}(\mathbf{R}) + \varphi^{(2)}(\mathbf{R}) + \ldots$$

where

$$\varphi^{(0)}(\mathbf{R}) = \frac{Q}{R}; \quad \varphi^{(1)}(\mathbf{R}) = \frac{(\mathbf{Rd})}{R^3}; \quad \varphi^{(2)}(\mathbf{R}) = \frac{X_i X_j Q_{ij}}{2!R^5}. \tag{1.28}$$

In Eqs. (1.28) the terms of different order in the small expansion parameter $r/R \ll 1$ can be interpreted as the total charge (zero moment), *dipole* (first) moment, and *quadrupole* (second) moment. These terms for the system of point charges are equal to

$$Q = \sum_a e_a, \quad \mathbf{d} = \sum_a e_a\mathbf{r}_a, \quad Q_{ij} = \sum_a e_a(3x_{ai}x_{aj} - \delta_{ij}\mathbf{r}_a^2). \tag{1.29}$$

Note that the trace of the quadrupole moment tensor is invariant and vanishes, $Q_{ii} = 0$.

For a system of extended charges, the sum should be replaced with an integration over the charge distribution density $\rho(\mathbf{r})$.

The analogous expansion in multipoles is also valid for the interaction energy of the system of charges with the inhomogeneous static electric field as long as the typical scale of the electric field inhomogeneity is much larger as the typical sizes of the charge system:

$$U = \sum_a \varphi(\mathbf{R} + \mathbf{r}_a) = U^{(0)} + U^{(1)} + U^{(2)} + \dots \tag{1.30}$$

where, correspondingly,

$$U^{(0)} = \sum_a e_a \varphi(\mathbf{R}) = Q\varphi(\mathbf{R}),$$
$$U^{(1)} = \frac{\partial \varphi(\mathbf{R})}{\partial \mathbf{R}} \sum_a e_a \mathbf{r}_a = -(\mathbf{Ed}), \quad U^{(2)} = \frac{1}{6} Q_{ij} \frac{\partial^2 \varphi(\mathbf{R})}{\partial X_i \partial X_j}. \tag{1.31}$$

The force being exerted on the system of charges is determined as $\mathbf{F} = -\boldsymbol{\nabla} U$ and, for a charge system having an electric dipole moment, it equals

$$\mathbf{F} = -\boldsymbol{\nabla}(U^{(0)} + U^{(1)}) = -q\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}(\mathbf{Ed}) = q\mathbf{E} + (\mathbf{d}\boldsymbol{\nabla})\mathbf{E}. \tag{1.32}$$

Here, we have taken into account the Maxwell equation curl $\mathbf{E} = 0$ for the static field.

The vector potential induced at the point $\mathbf{R}$ is equal to

$$\mathbf{A}(\mathbf{R}) = \int \frac{j(\mathbf{r})}{c|\mathbf{R} - \mathbf{r}|} \, d^3\mathbf{r} = \overline{\sum_a \frac{e_a \mathbf{v}_a}{c|\mathbf{R} - \mathbf{r}_a|}}$$

where the expressions are given for the system of the extended but spatially limited currents and moving point charges. The overscore (bar) means averaging in time since the current distribution is limited in space and therefore cannot be induced by charges moving with steady velocity $\mathbf{v}$.

When expanding the vector potential in multipole moments it is usually sufficient to restrict oneself to first or dipole terms which reads as

$$\mathbf{A}^{(1)}(\mathbf{R}) = \frac{1}{cR^3} \int (\mathbf{Rr})\mathbf{j}(\mathbf{r}) \, d^3\mathbf{r}$$

and can be expressed in terms of the *magnetic dipole moment* of the system $\boldsymbol{\mu}$:

$$\mathbf{A}(\mathbf{R}) = \frac{[\boldsymbol{\mu} \times \mathbf{R}]}{R^3} = -[\boldsymbol{\mu} \times \boldsymbol{\nabla}] \frac{1}{R} \, . \tag{1.33}$$

Here, the following notation is introduced:

$$\boldsymbol{\mu} = \frac{1}{2c} \int [\mathbf{r} \times \mathbf{j}] \, d^3\mathbf{r} = \overline{\sum_a \frac{e_a}{2c} [\mathbf{r}_a \times \mathbf{v}_a]}. \tag{1.34}$$

In the nonrelativistic case we have $\mathbf{v}_a = \mathbf{p}_a / m_a$ and thus the magnetic moment can be associated with the angular momentum of the particles:

$$\boldsymbol{\mu} = \sum_a \frac{e_a}{2m_a c} [\mathbf{r}_a \times \mathbf{p}_a] = \sum_a \frac{e_a}{2m_a c} \mathbf{M}_a \tag{1.35}$$

where in our case $\mathbf{M}_a$ is a time-independent angular momentum of a particle.

The quantity equal to

$$\gamma_a = \frac{e_a}{2m_a c} \tag{1.36}$$

is called the *gyromagnetic ratio* of a particle. Provided the charge-to-mass ratio is the same for all particles in the system, it becomes possible to introduce a single gyromagnetic ratio for the whole system equal to γ. Then, the magnetic moment is related to the *total angular momentum* as

$$\boldsymbol{\mu} = \frac{e}{2mc}\sum_a \mathbf{M}_a = \gamma\mathbf{M}. \tag{1.37}$$

In the external static inhomogeneous magnetic field the force exerted on the system of currents can also be expanded in multipole moments. As a rule, it is sufficient to consider only the interaction with the magnetic moment

$$\mathbf{F} = \frac{1}{c}\int [\mathbf{j}(\mathbf{r}) \times \mathbf{H}(\mathbf{R} + \mathbf{r})]\, d^3\mathbf{r} = \text{curl}\,[\mathbf{H} \times \boldsymbol{\mu}] = (\boldsymbol{\mu}\boldsymbol{\nabla})\mathbf{H}. \tag{1.38}$$

Here, we have taken into account that div $\mathbf{j} = 0$ in the static case and div $\mathbf{H} = 0$ according to the Maxwell equation. The force (1.38) can be obtained by introducing the effective energy U_m for the magnetic moment in the magnetic field:

$$U_m = -(\boldsymbol{\mu}\mathbf{H}), \quad \text{thus} \quad \mathbf{F} = -\boldsymbol{\nabla} U_m. \tag{1.39}$$

In addition, the system of currents is subjected to the torque $\mathbf{K}$ as well with

$$\mathbf{K} = \frac{1}{c}\int [\mathbf{r} \times [\mathbf{j} \times \mathbf{H}]]\, d^3\mathbf{r} = -[\mathbf{H} \times \boldsymbol{\mu}] = [\boldsymbol{\mu} \times \mathbf{H}].$$

Under this torque the angular momentum of the system changes in accordance with the *Larmor equation*

$$\frac{d\mathbf{M}}{dt} = \mathbf{K} = [\boldsymbol{\mu} \times \mathbf{H}]. \tag{1.40}$$

If the system of currents is induced by charges with the same gyromagnetic ratio γ, Eq. (1.40) reduces to

$$\frac{d\mathbf{M}}{dt} = \gamma[\mathbf{M} \times \mathbf{H}] = -[\boldsymbol{\Omega} \times \mathbf{M}] \tag{1.41}$$

where the *Larmor frequency* $\boldsymbol{\Omega} = \gamma\mathbf{H}$ is introduced. The vector $\boldsymbol{\Omega}$ is directed along or opposite to the magnetic field, depending on the sign of γ, i.e. the sign of the charge.

8 Free electromagnetic field

The general solution of the uniform wave equation can be given in the form of a plane wave, i.e. a function depending only on the argument $t - \mathbf{nr}/c$ where the unit vector $\mathbf{n}$ determines the direction of the field propagation. The general solution for the potentials is commonly treated in the Coulomb gauge. Then,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\dot{\mathbf{A}}; \quad \mathbf{H} = \text{curl}\ \mathbf{A} = -\frac{1}{c}[\mathbf{n} \times \dot{\mathbf{A}}] = [\mathbf{n} \times \mathbf{E}]. \tag{1.42}$$

The fields $\mathbf{E}$ and $\mathbf{H}$ lie in the plane normal to the unit vector $\mathbf{n}$ so that the three vectors $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{n}$ are mutually orthogonal and constitute a right-hand triple of vectors. The field proves to be *transverse*. The energy flux density in a plane electromagnetic wave is equal to

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{E} \times \mathbf{H}] = \frac{c}{4\pi}\mathbf{E}^2\mathbf{n} = \frac{c}{4\pi}\mathbf{H}^2\mathbf{n}.$$

The plane electromagnetic wave can be expanded into a Fourier integral:

$$\mathbf{A}(t,\mathbf{r}) = \int \mathbf{A}_{\mathbf{k},\omega} e^{i(\mathbf{kr}-\omega t)} \frac{d^3\mathbf{k} d\omega}{(2\pi)^4}, \quad \mathbf{A}_{\mathbf{k},\omega} = \int \mathbf{A}(t,\mathbf{r}) e^{-i(\mathbf{kr}-\omega t)}\, d^3\mathbf{r}\, dt. \tag{1.43}$$

The real part of the Fourier transform represents the specific case of a plane wave

$$\mathbf{A}(t,\mathbf{r}) = Re\{\mathbf{A}_{\mathbf{k},\omega} e^{i(\mathbf{kr}-\omega t)}\} \tag{1.44}$$

and is referred to as the *monochromatic plane wave*. The argument of such a wave, the wave *phase*, is $\omega t - \mathbf{kr} = \omega(t - \mathbf{nr}/c)$ and the vector $\mathbf{k}$ is called the wave vector with $|\mathbf{k}| = k = \omega/c = 2\pi/\lambda$, and $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$.

The wave phase can be represented in the Lorentz-invariant form $\omega t - \mathbf{kr} = k^\mu x_\mu$ by introducing the four-dimensional wave vector:

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right), \quad k^\mu k_\mu = \left(\frac{\omega}{c}\right)^2 - \mathbf{k}^2 = 0.$$

The electric and magnetic fields in the monochromatic plane wave are commonly written in the following form:

$$\mathbf{E}(t,\mathbf{r}) = Re\{\mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}\}, \quad \mathbf{H}(t,\mathbf{r}) = Re\{\mathbf{H}_0 e^{i(\mathbf{kr}-\omega t)}\}. \tag{1.45}$$

The electric field in the monochromatic plane wave can *always* be represented as

$$\begin{aligned}\mathbf{E} &= Re\{\mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}\} = Re\{(E_{01}\mathbf{e}^{(1)} + iE_{02}\mathbf{e}^{(2)}) e^{i(\mathbf{kr}-\omega t)}\} \\ &= \mathbf{e}^{(1)}|E_{01}|\cos(\mathbf{kr} - \omega t - \alpha) \mp \mathbf{e}^{(2)}|E_{02}|\sin(\mathbf{kr} - \omega t - \alpha)\end{aligned} \tag{1.46}$$

where $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are perpendicular unit vectors, maybe complex, and normal to the wave vector $\mathbf{k}$ as well. Thus, the monochromatic plane wave can be related to the *local basis* constituted by vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ which obey the following conditions:

$$(\mathbf{e}^{(1)*}\mathbf{e}^{(2)}) = (\mathbf{e}^{(1,2)}\mathbf{n}) = 0, \quad \mathbf{e}^{(2)} = [\mathbf{n} \times \mathbf{e}^{(1)}]. \tag{1.47}$$

The direction of the electric field vector in the local reference frame is called the *polarization* which can readily be represented as the unit vector $\mathbf{e} = \mathbf{E}_0/|\mathbf{E}_0| = e_1\mathbf{e}^{(1)} + e_2\mathbf{e}^{(2)}$. The monochromatic plane wave is *always polarized*. In the local reference frame the field vector $\mathbf{E}$ circumscribes an ellipse. If $E_{02} = \pm E_{01}$, the wave is circularly polarized. If $E_{02} = 0$, the wave is linearly polarized (see (1.46)).

Thus, the linear or circular polarization of a plane wave can be specified for the vectors of local basis if we choose the *real* vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ for linear polarization or $\mathbf{e}^{(\pm)} = (\mathbf{e}^{(1)} \pm i\mathbf{e}^{(2)})/\sqrt{2}$ for the circular one.

9 The retarded potentials and radiation

The system of charges can radiate an electromagnetic field and, correspondingly, lose radiation energy if the energy flux of the field across the infinitely distant closed surface is nonzero,

$$I = \oint (\mathbf{S}d\mathbf{f}) \neq 0. \tag{1.48}$$

To satisfy this, the radiation field must decay at large distances from the charge system according to

$$E,\ H|_{R\to\infty} \sim 1/R.$$

The quantity I, determining the energy losses per unit time, is called the *intensity of radiation*.

Such a distance dependence of the field can be obtained from expression (1.27) for the retarded potentials. At sufficiently large distances, the electric and magnetic radiation fields represent an outgoing spherical wave. The relation between the electric and magnetic fields is the same as in the plane wave, i.e. as in the free electromagnetic field for which the unit propagation vector $\mathbf{n}$ is directed from the radiating system to the point at which the field is observed.

In the case of a *nonrelativistic* system of charges, the retarded potentials can be expanded in multipoles. Here — in addition to the obvious small parameter $a/R \ll 1$ where a is the typical linear size of the system — one more parameter appears which is associated with the nonrelativistic type of motion, namely, $v/c \ll 1$ where v is the typical velocity of charge motion. Since the charges move within a finite volume, the typical time $T \sim a/v$ comes forward, resulting in the onset of the typical field frequency $\omega \sim c/a \gg v/a = T^{-1}$ in the expansion of the retarded potentials in monochromatic plane waves. This leads to a new dimensional quantity called the typical wavelength λ. Depending on the ratio between a, λ and R, one can distinguish three spatial regions for the fields obtained from the retarded potentials, as follows:

(1) *near-field* (or *quasi-static*) zone $a < R \ll \lambda$ in which retardation effects can be neglected;
(2) *transition* (*intermediate* or *induction*) zone $a \ll R \sim \lambda$ in which the retardation effect becomes essential but the radiation field is not yet formed;
(3) *far-field* (*radiation* or *wave*) zone $a \ll \lambda \ll R$ in which the radiation field becomes dominant, decaying as $1/R$ with distance.

Since the relation between the electric and the magnetic fields in the wave zone is analogue to that in the plane wave, it is convenient to choose the Coulomb gauge and consider the vector potential alone

$$\begin{aligned}\mathbf{A}(\mathbf{R},t) &= \frac{1}{cR}\int \mathbf{j}(\mathbf{r}', t - |\mathbf{R}-\mathbf{r}'|/c)\, d^3\mathbf{r}' \\ &= \frac{1}{cR}\int \left(\mathbf{j}(\mathbf{r}',t') + \frac{1}{c}\dot{\mathbf{j}}(\mathbf{r}',t')(\mathbf{r}'\mathbf{n}) + \frac{1}{2c^2}\ddot{\mathbf{j}}(\mathbf{r}',t')(\mathbf{r}'\mathbf{n})^2 + \dots \right) d^3\mathbf{r}' \qquad (1.49)\end{aligned}$$

where $t' = t - R/c$ is the retarded time synchronous for all charges in the nonrelativistic system.

The first term of the expansion in Eq (1.49), i.e.

$$\mathbf{A}_d(\mathbf{R},t) = \frac{1}{cR}\int \mathbf{j}(\mathbf{r}',t')\, d^3\mathbf{r}' = \left.\frac{\dot{\mathbf{d}}(t')}{cR}\right|_{t'=t-R/c} \qquad (1.50)$$

is determined by the *electric dipole moment* of the system. The far-field induced by potential (1.50) is called the *electric dipole radiation* and is equal to

$$\mathbf{H}_d(\mathbf{R},t) = \frac{[\ddot{\mathbf{d}}(t') \times \mathbf{n}]}{c^2R}, \quad \mathbf{E}_d(\mathbf{R},t) = \frac{[\mathbf{n} \times [\mathbf{n} \times \ddot{\mathbf{d}}(t')]]}{c^2R} \qquad (1.51)$$

where the right-hand sides of the equations in (1.51) should be evaluated at the retarded time $t' = t - R/c$. The angular intensity distribution of the electric dipole radiation is given by the formula

$$\frac{dI(t)}{d\Omega} = \frac{1}{4\pi c^2}\mathbf{H}^2 = \left.\frac{[\ddot{\mathbf{d}}(t') \times \mathbf{n}]^2}{4\pi c^3}\right|_{t'=t-R/c} \tag{1.52}$$

where $d\Omega$ is the solid angle element corresponding to the direction $\mathbf{n}$. The total intensity of the electric dipole radiation equals

$$I_d = \frac{2\ddot{\mathbf{d}}^2}{3c^3}. \tag{1.53}$$

The second term of expansion (1.49) straightforwardly yields the expression for the vector potential as a sum of two terms induced by the magnetic dipole and electric quadrupole moments $\mathbf{A}_2 = \mathbf{A}_m + \mathbf{A}_Q$, respectively. The term

$$\mathbf{A}_m(\mathbf{R}, t) = \frac{1}{cR}\frac{\partial}{\partial t}[\boldsymbol{\mu} \times \mathbf{n}] = \left.\frac{[\dot{\boldsymbol{\mu}}(t') \times \mathbf{n}]}{cR}\right|_{t'=t-R/c} \tag{1.54}$$

determines the magnetic dipole radiation. The second term gives the electric quadrupole radiation

$$A_{Q,i}(\mathbf{R}, t) = \left.\frac{n_j}{6c^2R}\frac{\partial^2}{\partial t^2}Q_{ij}(t')\right|_{t'=t-R/c} \quad \text{or}$$

$$\mathbf{A}_Q(\mathbf{R}, t) = \left.\frac{1}{6c^2R}\ddot{\mathbf{D}}(t')\right|_{t'=t-R/c}. \tag{1.55}$$

For convenience, vector $D_i = Q_{ij}n_j$ is commonly used.

The magnetic radiation field in second-order expansion is given by

$$\mathbf{H}_m + \mathbf{H}_Q = \frac{1}{c^2R}[[\ddot{\boldsymbol{\mu}}(t') \times \mathbf{n}] \times \mathbf{n}] + \frac{1}{6c^3R}[\dddot{\mathbf{D}}(t') \times \mathbf{n}].$$

On account of two terms in the above expression the angular distribution of the radiation intensity displays the interference or crossed term vanishing after integration over the total solid angle. As a result, the total radiation intensity in second-order expansion can be written as the sum

$$I_2 = I_m + I_Q = \frac{2}{3c^3}\ddot{\boldsymbol{\mu}}^2 + \frac{1}{180c^5}\dddot{Q}^2_{\alpha\beta}.$$

The radiation losses of the energy in the system of charged particles can be described as a work of some effective force called the *radiation damping force* or *Lorentz frictional force*. For example, for a single charged particle, one can write $I = (\mathbf{F}_{\rm rad}\mathbf{v})$ and, correspondingly, is lead to

$$\mathbf{F}_{\rm rad} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}} = \frac{2e}{3c^3}\dddot{\mathbf{d}}. \tag{1.56}$$

For an arbitrary system of charges radiating in the electric dipole approximation, the last term in Eq. (1.56) is valid as well.

In addition to the radiation damping force one can also introduce the torque resulting in the dissipative loss of angular momentum of the particle system

$$\overline{\frac{d\mathbf{M}}{dt}} = -\frac{2}{3c^3}\overline{[\dot{\mathbf{d}} \times \ddot{\mathbf{d}}]}.$$

10 Electromagnetic field of relativistic particles

It is not possible to describe the radiation of relativistically moving particles in terms of multipole moments. So, at first, we consider the radiation from a single relativistic particle. Next, employing the principle of superposition, we generalize the result to a system of particles. For a *single* particle, the partial solution for the 4-potential components (1.25) should be written with the aid of the retarded Green functions (1.26) in which the charge and current densities (1.6) correspond to a point particle with $\mathbf{r}_a = \mathbf{r}_0(t)$, $\mathbf{v} = \dot{\mathbf{r}}_0(t)$ being the particle velocity. For the vector potential, we have

$$\mathbf{A}(\mathbf{r}, t) = \frac{e}{c}\int \frac{\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}\mathbf{v}(t')\delta(\mathbf{r}' - \mathbf{r}(t'))\, d^3\mathbf{r}' dt'. \tag{1.57}$$

As a result of the calculations[5] we find that the vector potential generated by a relativistically moving charge is equal to

$$\mathbf{A}(\mathbf{r}, t) = \left.\frac{e\mathbf{v}}{cR(1 - (\mathbf{nv})/c)}\right|_{t'}. \tag{1.58}$$

The analogous expression for the scalar potential reads

$$\varphi(\mathbf{r}, t) = \left.\frac{e}{R(1 - (\mathbf{nv})/c)}\right|_{t'}. \tag{1.59}$$

The potentials (1.58) and (1.59) are called the *Liénard-Wiechert* potentials. In Eqs. (1.58) and (1.59) we have introduced the following notation: $\mathbf{R}(t') = \mathbf{r} - \mathbf{r}_0(t')$ and $R(t') = |\mathbf{R}(t')|$, $\mathbf{n}(t') = \mathbf{R}(t')/R(t')$ being the unit vector in the $\mathbf{R}$ direction. The right-hand sides in these expressions should be evaluated at the *retarded time* t' determined by the solution of the following equation:

$$t' - t + R(t')/c = 0. \tag{1.60}$$

The name "retarded time" is explained by the fact that the time difference $t - t' = R(t')/c$ is exactly equal to the time necessary to propagate the electromagnetic field from the point of its formation $\mathbf{r}_0(t')$ to the point $\mathbf{r}$ where the field is observed. Differentiating Eq. (1.60) with respect to the time t' gives the equation

$$\frac{dt}{dt'} = 1 - \frac{(\mathbf{nv})}{c} \tag{1.61}$$

which allows us to find a connection between the interval of observation time and the interval of retarded time.

The expressions for the magnetic and electric fields can be derived from the potentials (1.58) and (1.59):

$$\mathbf{H}(\mathbf{r},\ t) = \frac{e\,\{c\,[\mathbf{w}\times\mathbf{n}] + [\mathbf{n}\times[\,[\mathbf{v}\times\mathbf{w}]\times\mathbf{n}]\,]\}}{c^3R(1 - (\mathbf{nv})/c)^3} + \frac{e\,(1 - v^2/c^2)[\mathbf{v}\times\mathbf{n}]}{cR^2(1 - (\mathbf{nv})/c)^3}, \tag{1.62}$$

$$\mathbf{E}(\mathbf{r},\ t) = \frac{e\,[\mathbf{n}\times[(\mathbf{n} - \mathbf{v}/c)\times\mathbf{w}]\,]}{c^2R(1 - (\mathbf{nv})/c)^3} + \frac{e\,(1 - v^2/c^2)(\mathbf{n} - \mathbf{v}/c)}{R^2(1 - (\mathbf{nv})/c)^3} \tag{1.63}$$

5 For details, see the solution of problem 1.9.1.

where $\mathbf{w} = \dot{\mathbf{v}}(t')$ is the acceleration of a charge. The electromagnetic field, described by (1.62) and (1.63), always satisfies the condition $\mathbf{H}(\mathbf{r},\ t) = [\mathbf{n}(t') \times \mathbf{E}(\mathbf{r},\ t)]$, i.e. the vector $\mathbf{H}$ is perpendicular both to the vector $\mathbf{E}$ and to the unit vector connecting the charge position at the retarded time with the point of observation.

For large distances from the particle, the field in the wave zone is given by

$$\mathbf{E} = \frac{e\,[\mathbf{n} \times [(\mathbf{n} - \mathbf{v}/c) \times \mathbf{w}]]}{c^2 R(1 - (\mathbf{nv})/c)^3}, \qquad \mathbf{H} = [\mathbf{n} \times \mathbf{E}]. \tag{1.64}$$

Let the electromagnetic field energy $d\varepsilon$, emitted by the charge into the solid angle element $d\Omega$ in the direction $\mathbf{n} = \mathbf{R}/R$ during the time from t' to $t' + dt'$, pass across the surface element $dS = R^2 d\Omega$ during the time from t to $t + dt$. The moment of radiation t' and the moment of observation t are related by Eq. (1.60). At large $|\mathbf{r}| \gg |\mathbf{r}_0|$ distances from the radiating charge this energy in the wave zone is given by

$$d\varepsilon = \frac{c}{4\pi}|\mathbf{E}|^2 R^2 d\Omega\ dt$$

where $\mathbf{E}$ is determined by (1.64). Then, the angular distribution of the *radiation intensity*, i.e. the energy registered by an observer per unit time and solid angle, is equal to

$$\frac{dI}{d\Omega} = \frac{d^2\varepsilon}{d\Omega dt} = \frac{e^2}{4\pi c^3} \left. \frac{|[\mathbf{n} \times [(\mathbf{n} - \mathbf{v}/c) \times \mathbf{w}]]|^2}{(1 - (\mathbf{nv})/c)^6} \right|_{t'}. \tag{1.65}$$

The total radiation intensity $I = d\varepsilon/dt$ can be obtained from this angular distribution (1.65) by integration over the total solid angle.

Frequently, the *radiation power*, i.e. the energy emitted by a particle per unit retarded time t', becomes of interest. It is clear that the radiation power equals the rate of the particle energy losses, but with the opposite sign:

$$W = \frac{d\varepsilon}{dt'} = -\frac{d\mathcal{E}}{dt'}, \tag{1.66}$$

$\mathcal{E}$ being the particle energy, see (1.3). We will distinguish between these two quantities, namely radiation intensity I and radiation power W. Involving the condition $dt = dt'(1 - (\mathbf{nv})/c)$, which follows from Eq. (1.61), we obtain the relation between the angular distributions of radiation intensity I and radiation power W:

$$\frac{dW}{d\Omega} = \frac{d^2\varepsilon}{dt' d\Omega} = \left(1 - \frac{(\mathbf{nv})}{c}\right)\frac{dI}{d\Omega}. \tag{1.67}$$

We emphasize that, as can be seen from (1.67), the difference between power and intensity is significant in the relativistic case. If $v/c \ll 1$, this distinction becomes negligible.

Thus, the rate of total energy loss of a particle equals

$$\frac{d\mathcal{E}}{dt'} = -W = -\int \frac{e^2}{4\pi c^3} \frac{[\mathbf{n} \times [(\mathbf{n} - \mathbf{v}/c) \times \mathbf{w}]]^2}{(1 - (\mathbf{nv})/c)^5} d\Omega\ . \tag{1.68}$$

The direct integration over the angle in (1.68) is cumbersome. However, for calculating the total radiation power it is sufficient to see that the quantity $d\mathcal{E}/dt'$ is a relativistic invariant and, therefore, its expression can be written in the relativistic covariant representation

$$\frac{d\mathcal{E}}{dt'} = \frac{2}{3}e^2 c\ w^{\mu} w_{\mu} \tag{1.69}$$

where w^μ is the 4-acceleration of a particle, see (1.2). For the proper reference frame of a particle K_0 in which the instantaneous particle velocity vanishes $\mathbf{v}_0 = 0$, the expression (1.69) for the rate of energy loss reduces to

$$\left.\frac{d\mathcal{E}}{dt'}\right|_{K_0} \equiv \frac{d\mathcal{E}_0}{dt'_0} = -\frac{2e^2\mathbf{w}_0^2}{3c^3}$$

where $\mathbf{w}_0$ is the acceleration of a particle in its proper or instantaneous comoving reference frame. The obtained equation is called the *Larmor formula*. Since in the laboratory reference frame the velocity 4-vector at time t' reads according to (1.2) as

$$u^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{cd\tau} = \frac{1}{c\sqrt{1-v^2/c^2}}\frac{dx^\mu}{dt'} \equiv \frac{\gamma}{c}\frac{d}{dt'}(ct', \mathbf{r}) = \gamma\left(1,\ \frac{\mathbf{v}}{c}\right),$$

the components of the 4-acceleration will be equal to

$$w^\mu = \frac{du^\mu}{ds} \equiv \frac{\gamma}{c}\frac{d}{dt'}\left(\gamma,\ \gamma\frac{\mathbf{v}}{c}\right) = \frac{\gamma^2}{c^2}\left(\frac{\gamma^2}{c}\mathbf{vw},\ \mathbf{w} + \frac{\gamma^2}{c^2}(\mathbf{vw})\mathbf{v}\right). \tag{1.70}$$

Substituting the result obtained into Eq. (1.69), we arrive at the *Liénard formula* generalizing the above Larmor formula:

$$\frac{d\mathcal{E}}{dt'} = -\frac{2e^2\gamma^6}{3c^3}\left\{\mathbf{w}^2 - \frac{[\mathbf{v}\times\mathbf{w}]^2}{c^2}\right\}. \tag{1.71}$$

11 The scattering of electromagnetic waves

Under influence of the field of the electromagnetic wave incident onto the system of charges the charges will be set in some accelerated motion which, correspondingly, results in the radiation of electromagnetic waves. On the other hand, the field of the incident electromagnetic wave performs work on the system of charges and, therefore, wave energy is absorbed. Thus, the electromagnetic field emitted by the charge system in all directions under influence of the incident wave can be treated as the field of a *scattered wave*. All scattering processes are commonly described in terms of the *differential scattering cross section*

$$d\,\sigma = \frac{\overline{dI}}{\overline{|\mathbf{S}|}}$$

where $\overline{dI}$ is the intensity of radiation outgoing from the charge system in the given direction towards the solid angle element $d\Omega$ per unit time and $\overline{|\mathbf{S}|}$ is the energy flux density of the wave incident on a unit area of the charge system per unit time.

The scattering of a monochromatic plane wave is usually treated in order to obtain the *spectral expansion* of the differential cross section of scattering. In the monochromatic plane wave the Poynting vector is given by

$$\overline{|\mathbf{S}|} = \frac{c}{4\pi}\overline{(Re\{\mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}\})^2} = \frac{c}{8\pi}(E_{01}^2 + E_{02}^2).$$

The scattering at the system of charges is described readily in the nonrelativistic approximation. For a single charge, it is sufficient to consider electric dipole radiation, hence

$$\frac{d\,I}{d\,\Omega} = \frac{e^2}{4\pi c^3}[\ddot{\mathbf{r}}\times\mathbf{n}]^2.$$

Here, the acceleration $\ddot{\mathbf{r}}$ is determined by the equation of motion in the electric field of the incident wave since the effect of the magnetic field can be neglected in the nonrelativistic case and the electric field of the plane wave can be treated as uniform within the same accuracy:

$$m\ddot{\mathbf{r}} = e\mathbf{E} + \mathbf{F}_{\mathrm{rad}}, \quad \text{or} \quad \ddot{\mathbf{r}} - \frac{2e^2}{3mc^3}\dddot{\mathbf{r}} = \frac{e}{m}Re\{e\mathbf{E}_0 e^{-i\omega t}\}. \tag{1.72}$$

The partial solution for the equation of charge motion reads

$$\mathbf{r} = \frac{e^2}{m\omega^2}Re\left\{\frac{1}{1+i\gamma}\ \mathbf{E}_0 e^{-i\omega t}\right\} \quad \text{where} \quad \gamma = \frac{2e^2}{3mc^3}\omega.$$

Here, γ is referred to as the *natural linewidth* of radiation. The expression for the differential cross section of scattering, provided $\gamma \ll 1$, is given by

$$\frac{d\,\sigma}{d\,\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{[\mathbf{e}^{(1)} \times \mathbf{n}]^2 + (E_{01}/E_{02})^2[\mathbf{e}^{(2)} \times \mathbf{n}]^2}{1 + (E_{01}/E_{02})^2}. \tag{1.73}$$

The total scattering cross section is found by integrating the differential cross section (1.73) over solid angle and equals

$$\sigma_T = \frac{8\pi}{3}r_e^2 \tag{1.74}$$

where $r_e = e^2/mc^2$ is the *classical electron radius*. The total cross section (1.74) is called the *Thomson cross section*.

The scattering of the wave at a free particle is accompanied by the appearance of some wave-induced *force* acting on the scattering particle. The time-averaged force exerted on the particle is determined by the average value of the field momentum absorbed by the particle per unit time. The expression for the force can be represented as

$$\overline{\mathbf{F}} = \sigma_T \overline{w}\mathbf{n} \tag{1.75}$$

where σ_T is the Thomson cross section (1.74), $\mathbf{n}$ is the unit vector in the propagation direction of the incident wave, and $\overline{w} = \overline{E^2}/4\pi$ is the average energy density of the field. Note that the momentum of the field equals the field energy divided by the speed of light.

The polarization properties of the plane wave can conveniently be described with the aid of the second-rank tensor $\rho_{\alpha\beta}$ composed of the components of the complex unit vector $\mathbf{e}$. Since this vector $\mathbf{e}$ has only two components in the plane normal to the direction of wave propagation, let us denote them with Greek subscripts as

$$\rho_{\alpha\beta} = e_\alpha^* e_\beta; \quad \rho_{\alpha\beta}^* = \rho_{\beta\alpha}\ .$$

Thus, the polarization tensor is a 2×2 self-adjoint matrix and its trace satisfies

$$\mathrm{Tr}\ \hat{\rho} = \rho_{\alpha\alpha} = |e_1|^2 + |e_2|^2 = 1.$$

For the linearly polarized wave, the polarization tensor is quite simple. The only nonzero element of the matrix is on the diagonal, either ρ_{11} or ρ_{22}. Provided the wave has circular polarization, one has $\rho_{11} = \rho_{22} = 1/2$ and $\rho_{12} = \pm i/2$ where the *plus* sign corresponds to right-handed and the *minus* sign to left-handed circular polarization. The determinant of matrix $\hat{\rho}$ vanishes, i.e. det $\hat{\rho} = 0$. The polarization tensor contains the full information on the polarization properties of the electromagnetic wave and is widely used for describing both monochromatic and natural radiation.

Problems

1.1 Vectors and tensors in Euclidean space

1.1.1 Show that the scalar product of two real vectors is invariant with respect to coordinate transformations.

1.1.2 Show that the bilinear combination A_iB_j formed by the product of the components of two vectors A_i and B_j is a second-rank tensor.

1.1.3 Show that the tensor δ_{ij} can be represented as

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j}.$$

1.1.4 Verify that the tensor δ_{ij} is invariant with respect to orthogonal coordinate transformations.

1.1.5 Calculate the contraction δ_{ii}.

1.1.6 Calculate $\delta_{ij}\delta_{ik}\delta_{kl}\delta_{li}$.

1.1.7 Calculate $e_{ijk}e_{mnk}$.

1.1.8 Calculate $e_{ijk}e_{mjk}$.

1.1.9 Calculate $e_{ijk}e_{ijk}$.

1.1.10 Calculate $e_{ijk}e_{jik}$.

1.1.11 Transform the expression curl curl $\mathbf{a}$.

1.1.12 Transform the expression div $[\mathbf{a} \times \mathbf{b}]$.

1.1.13 Transform the expression curl $[\mathbf{a} \times \mathbf{b}]$.

1.1.14 Transform the expression grad $(\mathbf{ab})$.

1.1.15 Assuming the vector $\boldsymbol{\mu}$ to be constant, calculate div $(\boldsymbol{\mu} r)$.

1.1.16 Calculate grad $(\boldsymbol{\mu}\mathbf{r})$.

1.1.17 Calculate curl $[\boldsymbol{\mu} \times \mathbf{r}]$.

1.1.18 Calculate div $(\boldsymbol{\mu} f(r))$.

1.1.19 Calculate curl $(\boldsymbol{\mu} f(r))$.

1.1.20 Calculate the gradient of the exponential term $e^{i\mathbf{kr}}$.

1.1.21 Assuming vector $\mathbf{A}$ to be constant, calculate div$(\mathbf{A}e^{i\mathbf{kr}})$.

1.1.22 Calculate curl$(\mathbf{A}e^{i\mathbf{kr}})$.

1.1.23 Using the properties of the tensor e_{ijk}, show that a cross product is anticommutative, i.e. antisymmetric with respect to interchanging the factors.

1.1.24 Using the properties of the tensor e_{ijk}, derive the formula of the vector triple product expansion, or Lagrange's formula with the mnemonic **bac − cab**

$$[\mathbf{a} \times [\mathbf{b} \times \mathbf{c}]] = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab}).$$

1.1.25 Calculate $(\mathbf{a}\boldsymbol{\nabla})\mathbf{r}$.

1.1.26 Transform the integral $\int_V f(\mathbf{r})\text{div}\,\mathbf{A}(\mathbf{r})\ dV$.
Use the invariant and tensor forms of representation.

1.1.27 Using the invariant and tensor forms of representation, transform the volume integral of the integrand (grad$f$ curl $\mathbf{A}$) to a surface integral.

1.1.28 For this and the next expressions, calculate the surface integrals, connecting them with the volume integrals $\oint_S(\mathbf{r}\ d\mathbf{S})$.

1.1.29 Calculate $\oint_S z\ dx\ dy$.

1.1.30 Assuming the vector $\mathbf{c}$ to be constant, calculate $\oint_S \mathbf{r}(\mathbf{c}\ d\mathbf{S})$.

1.1.31 Assuming the vector $\mathbf{c}$ to be constant, calculate $\oint_S(\mathbf{cr})\ d\mathbf{S}$.

1.1.32 Transform the surface integral $\oint_S[d\mathbf{S} \times \mathbf{A}]$ to the volume one.

1.1.33 Assuming the vector $\mathbf{c}$ to be constant, calculate $\oint_S(\mathbf{c}d\mathbf{S})\mathbf{A}$.

1.1.34 Transform the contour integral $\oint_L d\ d\mathbf{l}$ with the scalar function f as integrand to the surface integral

$$\oint_L f\ d\mathbf{l}.$$

1.1.35 Prove the formula

$$\frac{1}{2}\oint_L [d\mathbf{r} \times \mathbf{r}] = \int_S d\mathbf{S}.$$

1.2 Vectors and tensors in Minkowski space

1.2.1 Calculate $e^{\mu\nu\kappa\lambda}e_{\mu\nu\kappa\lambda}$.

1.2.2 Calculate $e^{\mu\nu\kappa\lambda}e_{\rho\nu\kappa\lambda}$.

1.2.3 Calculate $e^{\mu\nu\kappa\lambda}e_{\rho\sigma\kappa\lambda}$.

1.2.4 Calculate a scalar $\tilde{e}$ dual to the completely antisymmteric tensor $e_{\mu\nu\kappa\lambda}$ in four-dimensional Minkowski space.

1.2.5 Express the twice dual tensor $\tilde{\tilde{F}}_{\mu\nu}$ in terms of the initial antisymmetric tensor $F_{\mu\nu}$.

1.2.6 Assuming $G_{\mu\nu\lambda} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}$, write $\tilde{G}^\lambda$ via $\tilde{F}^{\lambda\mu}$.

1.2.7 Assuming $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, write $\tilde{F}^{\nu\lambda}$ via $\tilde{A}^{\nu\lambda\mu}$.

1.2.8 Express $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$ via the antisymmetric tensor $F_{\mu\nu}$.

1.3 Relativistic kinematics

1.3.1 Write a product of two successive Lorentz transformations for velocities $v_1/c = \tan\theta_1$ and $v_2/c = \tan\theta_2$ in the direction of the x axis. Does the resultant transformation correspond to the Lorentz transformation? If so, what velocity $v/c = \tan\theta$ does the resultant transformation correspond to?

1.3.2 Show that the matrix of a boost, i.e. rotation-free Lorentz transformation, in the direction of the x axis can be represented as a matrix exponential $\Lambda_x(\theta) = \exp(\theta a_x)$ where θ is the rapidity $v/c = \tan\theta$ and a_x is an unknown matrix to be called the *boost generator* in the x axis direction. Find explicitly the matrix a_x. Write the boost matrices and generators in the direction of the y and z axes.

1.3.3 Show that the matrix of rotation around the x axis by the angle φ, $\mathrm{R}_x(\varphi)$, can be represented as a matrix exponential $\mathrm{R}_x(\varphi) = \exp(\varphi\ b_x)$ where b_x is an unknown matrix called the *rotation generator* around the x axis. Find this matrix b_x in explicit form. Write the matrices and generators of rotation around the y and z axes.

1.3.4 Show that the volume element $d^4x = cdt\ dx\ dy\ dz$ is invariant under rotations, Lorentz boosts and their combinations.

1.3.5 The origin of the reference frame K' moves at velocity $\mathbf{v}$ with respect to frame K and the coordinate axes in K' are at the same angles with $\mathbf{v} = (v_x,\ v_y)$ as the coordinate angles do in frame K. Write the Lorentz transformation from frame K to frame K'.

1.3.6 Determine the position of axes (x', y') of frame K' (see problem 1.3.5) in the frame K at zero time $t = 0$ according to a clock in frame K.

1.3.7 Let velocities in the reference frames K_1, K_2, and K' lie in the (x, y) plane of laboratory frame K. The velocity of frame K_1 with respect to frame K equals $\mathbf{v}_1 = (V_x, 0, 0)$, and the velocity of frame K' with respect to frame K equals $\mathbf{V} = (V_x, V_y, 0)$. At what velocity $\mathbf{v_2} = (V_{2x}, V_{2y}, 0)$ should the frame K_2 move with respect to frame K_1 so that the frame K_2 be motionless with respect to frame K', i.e. its velocity with respect to laboratory frame K be equal to the velocity of frame K' with respect to K?

1.3.8 At what angle are the coordinate axes of frame K_2 rotated with respect to the axes of frame K'? (See Prs. 1.3.5 and 1.3.7.)

1.3.9 A force acts on a particle moving at velocity $\mathbf{v}$ and delivers the acceleration of magnitude $\dot{\mathbf{v}}$ to the particle. Determine the angular velocity at which the spin of a particle rotates with respect to the laboratory frame if the force acting on the particle does not affect the particle spin. The phenomenon, called the *Thomas precession*, results from the fact that a composition of two noncollinear Lorentz boosts does not reduce to a boost alone but represents a combination of boosts and rotations.

1.3.10 Two sources moving with the same velocity v along one straight line emit light pulses at one time in the frame of sources. The distance between the sources in the frame of sources equals l.
What is the time lapse for the light pulses to be registered with a receiver located on the same line as the sources but in front of them?

1.3.11 A plane mirror moves at velocity v in the direction of its normal. The monochromatic wave of frequency ω_1 strikes the mirror at angle θ_1 with respect to the normal. Find the direction and frequency of the reflected wave, assuming that the standard reflection law holds for the mirror at rest. *Hint*: introduce the wave and normal 4-vectors.

1.3.12 Choose a uniformly rotating reference frame, i.e. the frame of an observer who travels uniformly round the circumference of a rotating table. How does the metric tensor in such a reference frame look like?
Let one of two twins remain at rest and the other travel around the circumference. Which one of the twins will be younger when they will meet each other?

1.3.13 Determine the phase difference for propagating a light ray of frequency ω clockwise and counterclockwise in a rotating circular waveguide with radius R and refractive index n. The phenomenon is called the *Sagnac effect* and used for determining angular rotation velocity with *laser gyroscopes*.

1.3.14 An observer sits on a plane having a slit of width l and sees that a rod rushes at velocity (V_x, V_y) in the direction of the slit. The rod is parallel to the plane and has length $L > l$, the x axis being parallel to the plane and the y axis being perpendicular to it. The velocity of the rod is such that in the observer's frame the rod has a length slightly *less* than l. Will the rod pass safely through the slit if we take into account that it is the slit width that contracts in the frame of the rod? Which end of the rod will pass first through the slit from the viewpoints of the observer and the rod?

1.3.15 Two particles of masses m_1 and m_2 have momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. Determine:
(1) the velocity of the center-of-inertia frame;
(2) the energy in the center-of-inertia frame;
(3) the energy of the second particle in the rest frame of the first particle;
(4) the relative velocity of the particles.

1.3.16 In an accelerator a beam of particles of mass m and energy $\mathcal{E} \gg mc^2$ collides with a fixed target composed of particles with the same mass. Determine the effective mass for a pair of particles from the beam and the target. Find the velocity of the center-of-inertia frame for a pair of particles. As an example, consider the collision of accelerated protons ($mc^2 = 1\,\text{GeV}$) with the protons in the fixed target at the following energies of incident protons:
(1) $\mathcal{E} = 70\,\text{GeV}$ (Serpukhov accelerator);
(2) $\mathcal{E} = 1\,\text{TeV}$ (Tevatron at Fermilab, Batavia, USA);
(3) $\mathcal{E} = 7\,\text{TeV}$ (Large Hadron Collider, CERN, Geneva).

1.3.17 A collider provides head-on collisions for beams of particles with mass m_1 and energy $\mathcal{E}_1 \gg m_1c^2$ and with mass m_2 and energy $\mathcal{E}_2 \gg m_2c^2$. Determine the effective mass of a pair of particles from the first and second beams, and the center-of-inertia velocity of a pair. Give the numerical answer for head-on collisions of protons with energy $\mathcal{E}_1 = 800\,\text{GeV}$ and electrons with energy $\mathcal{E}_2 = 30\,\text{GeV}$ (accelerator HERA, Germany).

1.3.18 What energy $\mathcal{E}'$ should be achieved in a fixed-target accelerator in order to obtain the same center-of-inertia energy as for the 7 TeV collider (Large Hadron Collider)? For an estimate, take $m_1c^2 = m_2c^2 = 1\,\text{GeV}$.

1.3.19 Show that the energy ε and momentum projection p_x onto the x axis can be parameterized for a particle of mass m as $\varepsilon = m_\perp \cosh\alpha$, $p_x = m_\perp \sinh\alpha$ where $m_\perp^2 = m^2 + p_y^2 + p_z^2$ and $\tanh\alpha = v_x$. (Here, we put $c = 1$.)

1.3.20 A π^0-meson runs the distance L and decays into two γ-quanta emitted at angles α and β with respect to the particle's velocity. Determine the lifetime of a π^0-meson in its rest frame.

1.3.21 Determine the mass m of a particle if it decays into two particles with masses m_1 and m_2. The momenta of these particles, $\mathbf{p}_1$ and $\mathbf{p}_2$, are known from experiment.

1.3.22 Determine the mass m_1 of a particle if you know that it represents one of two particles created in the decay of a particle with mass m and momentum $\mathbf{p}$. The mass m_2 and momentum $\mathbf{p}_2$ of the second particle are known from experiment.

1.3.23 A π^0-meson flying at velocity $\mathbf{v}$ decays into two γ-quanta. Determine the minimum angle of separation of the γ-quanta, θ_{min}.

1.3.24 An ultrarelativistic particle with mass m and energy ε_0 scatters elastically on a fixed nucleus with mass $M \gg m$. Determine the behavior of the final particle energy ε as a function of the scattering angle θ.

1.3.25 A particle with mass m_1 and velocity $\mathbf{v}$ collides with a particle of mass m_2 which is at rest and is absorbed by the latter. Find the mass M and velocity $\mathbf{V}$ of the new particle.

1.3.26 An ultrarelativistic particle with mass m and energy ε_0 scatters on a fixed nucleus of mass $M \gg m$ and excites the latter. The energy of the excited nucleus $\Delta\mathcal{E}$ in its rest frame satisfies the inequality $mc^2 \ll \Delta\mathcal{E} \ll Mc^2$. Find the behavior of the final particle energy ε as a function of the scattering angle θ.

1.3.27 A photon of frequency ω_0 scatters at a freely moving electron with momentum $\mathbf{p}_0$ at an angle θ_0 with respect to the direction in which the photon propagates (*Compton scattering*). Find the behavior for the frequency ω of the scattered photon as a function of the direction of its propagation.

1.3.28 A photon of energy $\varepsilon = 2\,\text{eV}$ scatters at an ultrarelativistic electron flying with energy $\mathcal{E} = 200$ GeV $\gg \varepsilon$ in the opposite direction. Find the maximum energy of the scattered photon. For what relation between $\mathcal{E}$, ε and the electron mass m can this be achieved?

1.3.29 For the neutrinos appearing in the decay of a 6 GeV π-mesons, determine their energy spectrum, maximum and average energies, and angular distribution if the decay of π-meson, $\pi \to \mu + \nu$, is known to be isotropic in its rest frame. The mass of the π-meson is $m_\pi c^2 \approx 140\,\text{MeV}$, the mass of the muon is $m_\mu c^2 \approx 105\,\text{MeV}$, and the neutrino mass[6] is $m_\nu c^2 \approx 0$. Find the characteristic interval of the neutrino emission angles. Plot the energy and angular distributions for the neutrinos emergent in the π-meson decay.

1.3.30 A π^0-meson flying at velocity $\mathbf{v}$ decays into two γ-quanta. Find the angular distribution $dN/Nd\Omega$ for γ-quanta in the laboratory reference frame if the distribution is spherically symmetric in the π^0-meson rest frame.

1.3.31 For the K-meson decay, $K \to \pi + e + \overline{\nu}$, determine the maximum possible energy for each of three emerging particles. Use the values $m_K c^2 = 500\,\text{MeV}$, $m_e c^2 = 0.5\,\text{MeV}$, $m_\pi c^2 = 135\,\text{MeV}$, and $m_\nu = 0$.

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6 According to modern experimental data, the neutrino mass is very small but nonzero, namely $m_\nu c^2 \simeq 1\text{eV}$ or even less. Therefore, in the ultrarelativistic case $\mathcal{E}_\nu \gg m_\nu c^2$ one can assume with good accuracy that $m_\nu c^2 \approx 0$. See also Pr. 1.3.31.

1.3.32 An accelerated proton collides with a proton of the target at rest. The reaction results in the production of particle a and antiparticle $\tilde{a}$: $p + p \to p + p + a + \tilde{a}$. Find the energy threshold for this reaction if particle a is as follows: (a) proton p of mass about 1 GeV; (b) deuteron d of mass about 2 GeV; (c) α-particle or ^{4}He nucleus of mass about 4 GeV.

1.3.33 A collider provides for the reaction $e^+ + e^- \to \mu^+ + \mu^-$. Assuming the electron and positron energies to be known, find the energy and momentum of muons. What is the energy threshold for this reaction? Compare with the energy threshold in the case when the accelerated positrons collide with the electrons at rest. The electron mass is $m_e c^2 = 0.5$ MeV and the muon mass is $m_\mu c^2 = 100$ MeV.

1.3.34 A proton with energy $\mathcal{E} = 3$ GeV, scattering on a proton of the target, transfers to it an energy $\varepsilon = 1$ MeV. Determine the scattering angle in the center-of-inertia system and in the laboratory reference frame.

1.3.35 What is, in the ultrarelativistic limit, the connection between the scattering angle for a collision of two identical particles of mass m in a fixed-target system and in the center-of-inertia system? The energy of the incident particle in the fixed-target frame equals $\mathcal{E}$.

1.3.36 A free nucleus of mass M has an excited state whose energy exceeds the ground state energy by ε.

(1) The excited nucleus emits a γ-quantum and relaxes to the ground state. Find the energy of this γ-quantum.
(2) Determine the recoil energy which the nucleus receives.
(3) Determine the minimum velocity at which the excited nucleus should move in order to excite a nucleus in the ground state at rest by the γ-quantum emitted by the moving excited nucleus.

1.3.37 Determine the maximum energy of neutrinos produced in the β^+-decay of the nuclei

$$^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \nu_e \quad \text{and} \quad {}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \nu_e \,,$$

assuming that protons are distributed with a homogeneous density over the nucleus volume. The nucleus's size is $R_0 = 1.5 \times 10^{-13} A^{1/3}$cm, A being the mass number. For simplicity, use the same binding energies for protons and neutrons. The difference in the neutron and proton masses is $m_n c^2 - m_p c^2 = 1.29$ MeV.

1.4 The Maxwell equations

1.4.1 Derive the wave equations for the electromagnetic field from the Maxwell equations.

1.4.2 Two Maxwell equations allow us to calculate the time derivatives of the fields **E** and **H** provided these fields are given on some hyperplane $t = \text{const}$:

$$\frac{\partial \mathbf{H}}{\partial t} = -c \operatorname{curl} \mathbf{E}, \qquad \frac{\partial \mathbf{E}}{\partial t} = c \operatorname{curl} \mathbf{H} - 4\pi \mathbf{j}. \tag{1}$$

The other two Maxwell equations

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{div} \mathbf{E} = 4\pi\rho \tag{2}$$

do not contain time derivatives and, therefore, their validity can be checked for a separate hyperplane $t = t_0$.

Let Eqs. (2) be checked for the initial conditions given at some time $t = t_0$. Next, the fields **E** and **H** are calculated for arbitrary time moments with the aid of Eqs. (1).

Under what condition will Eqs. (2) be satisfied for an arbitrary point in time t different from t_0?

1.4.3 Modify the first pair of the Maxwell equations by including the density of hypothetical magnetic monopoles or magnetic charges ρ_m and the current density of these magnetic monopoles $\mathbf{j}_m$ in such a way that the continuity equation is satisfied for them. Solve problem 1.4.2 for the modified equations.

1.4.4 Rewrite the Maxwell equations in terms of the complex vector field $\mathbf{F} = \mathbf{E} + i\mathbf{H}$. How does such a form of the Maxwell equations change in the presence of magnetic monopoles? See Pr. 1.4.3.

1.4.5 Substituting the expressions for the fields **E** and **H** via the potentials φ and **A** into the Maxwell equations in three-dimensional form, derive the field equations for the potentials.

What form does the first pair of the Maxwell equations reduce to?

1.4.6 Substituting the expressions for the fields $F_{\mu\nu}$ via the potentials A_μ into the Maxwell equations in four-dimensional form. Derive the field equations for the potentials.

What form does the first pair of the Maxwell equations reduce to?

1.4.7 What form will the field equations for the potentials take when the *Lorentz gauge condition* $\partial_\mu A^\mu = 0$ or $\frac{1}{c}\frac{\partial \varphi}{\partial t} + \operatorname{div} \mathbf{A} = 0$ is imposed? Consider both three- and four-dimensional notations.

1.4.8 What form will the field equations for the potentials take in three-dimensional notation when the *Coulomb gauge condition* $\operatorname{div} \mathbf{A} = 0$ is imposed?

1.4.9 What form will the field equations for the potentials take in three-dimensional notation when the *Weyl gauge condition* $\varphi = 0$ is imposed?

1.4.10 For gauge $\varphi = 0$, parametrize the constant uniform electromagnetic field of the general form with the aid of the vector potential **A**, the fields **E** and **H** being two arbitrary constant vectors.

1.4.11 For potentials of the general form $A^\mu = (\varphi, \mathbf{A})$, find the gauge transformations which satisfy the following gauge conditions:

(1) $\partial_\mu A^\mu = 0$, i.e. $\frac{1}{c}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0$ (*Lorentz gauge*),

(2) $\operatorname{div}\mathbf{A} = 0$ (*Coulomb gauge*),

(3) $\varphi = 0$ (*Weyl gauge*).

1.4.12 Find the form of the residual gauge transformation or constraint on function f for each of the following gauges:

(1) $\partial_\mu A^\mu = 0$, i.e. $\frac{1}{c}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0$ (*Lorentz gauge*),

(2) $\operatorname{div}\mathbf{A} = 0$ (*Coulomb gauge*),

(3) $\varphi = 0$ (*Weyl gauge*).

1.4.13 Show that the continuity equation for charges and currents results from the invariance of the electromagnetic field action with respect to gauge transformations.

1.4.14 The action function for the electromagnetic field (1.12) reads

$$S_{\text{field}}[\underline{A}(\underline{x})] = -\frac{1}{16\pi c}\int F^{\mu\nu}F_{\mu\nu}\, d^4x = \frac{1}{c}\int \mathcal{L}\, d^4x.$$

Obtain the energy-momentum tensor $\tilde{T}^{\mu\nu}$ according to Eq. (1.14). Is this tensor gauge-invariant?

1.4.15 Find an additive $\delta T^{\mu\nu}$ to the energy-momentum tensor from Pr. 1.4.14 such that the new energy-momentum tensor $T^{\mu\nu} = \tilde{T}^{\mu\nu} + \delta T^{\mu\nu}$ would be symmetric and $\partial_\mu \delta T^{\mu\nu} = 0$ under the condition that the Maxwell equations are satisfied without sources. Is the new tensor $T^{\mu\nu}$ gauge-invariant?
Verify that $\partial_\mu T^{\mu\nu} = 0$ in the absence of field sources.

1.4.16 Calculate $\partial_\mu T^{\mu\nu}$ for the tensor from Pr. 1.4.15 in the presence of charges and currents. Derive the equations for the energy-momentum conservation law.

1.4.17 Find the trace of the energy-momentum tensor T^μ_μ for the electromagnetic field.

1.4.18 Find the pressure on one of the plates of a vacuum-separated thin capacitor if you start from the following assumptions:

(1) the force density $\rho\mathbf{E}$ is known;

(2) the drop of the electric field pressure is known.

1.4.19 Find the pressure on the wall of a coreless solenoid under the following assumptions:

(1) the force density $[\mathbf{j}\times\mathbf{H}]/c$ is known;

(2) the drop of the magnetic field pressure is known.

1.5 The motion of a charged particle in an external field

1.5.1 For a relativistic particle, prove that the force power ($\mathbf{vF}$) equals the energy variation rate.

1.5.2 Consider the special features of the relativistic motion for a particle of charge e and mass m in uniform crossed fields $\mathbf{E}$ and $\mathbf{H}$, the angle between the fields being equal to θ.

1.5.3 Determine the coordinates $\mathbf{r}$ and time t of a charged particle in the laboratory reference frame as a function of proper time τ for particle motion in a uniform constant electric field $\mathbf{E}$. At the initial time the particle is at the frame origin and has the energy $\mathcal{E}_0$ and momentum $\mathbf{p}_0$.

1.5.4 Determine the coordinates $\mathbf{r}$ and time t of a charged particle in the laboratory reference frame as a function of proper time τ for particle motion in a uniform constant magnetic field $\mathbf{H}$. At the initial time the particle is at the frame origin and has the energy $\mathcal{E}_0$ and momentum $\mathbf{p}_0$.

1.5.5 Determine the coordinates $\mathbf{r}$ and time t of a charged particle in the laboratory reference frame as a function of proper time τ for a particle moving in uniform constant electric and magnetic fields perpendicular to each other. At the initial time the particle is at the frame origin and has the energy $\mathcal{E}_0$ and momentum $\mathbf{p}_0$.

1.5.6 Determine the trajectory of motion of a charged particle in the crossed electric and magnetic fields, $\mathbf{E} \perp \mathbf{H}$, of the same magnitudes.

1.5.7 Find the kinetic energy of a particle $T = \mathcal{E} - mc^2$ as a function of proper time τ when it moves in constant uniform electric and magnetic fields parallel to each other, $\mathbf{E}||\mathbf{H}$. At the initial time the particle is at rest at the frame origin.

1.5.8 Determine the coordinates $\mathbf{r}$ and time t of a charged particle in the laboratory reference frame as a function of proper time τ when it moves in constant uniform electric and magnetic fields parallel to each other. At the initial time the particle is at the frame origin and has the energy $\mathcal{E}_0$ and momentum $\mathbf{p}_0$.

1.5.9 Find the kinetic energy of a particle $T = \mathcal{E} - mc^2$ as a function of proper time τ when it moves in crossed uniform constant electric and magnetic fields, $\mathbf{E} \perp \mathbf{H}$. At the initial time the particle is at rest at the frame origin.

1.5.10 Find the deceleration time t_{br} and the stopping distance l for a relativistic particle of charge e, mass m and initial energy $\mathcal{E}_0$ in an decelerating uniform electric field $\mathbf{E}$ parallel to the initial velocity of a particle.

1.5.11 A beam of π^+-mesons with the initial momentum $p_0 = 200\,\text{MeV}/c$ is injected into a linear accelerator. What is the accelerating field strength E necessary for accelerating, at least, half of the pions to the energy $\mathcal{E} = 200\,\text{GeV}$? What length should the accelerator have?
The π^+-meson mass is $mc^2 = 140\,\text{MeV}$. The π^+-meson lifetime is $\tau_{\pi^+} = 2.6 \times 10^{-8}\text{s}$ and the half-life for the decay $\pi^+ \to \mu^+ + \nu_\mu$ is $\tau_{1/2} = \tau_{\pi^+} \ln 2$.

1.5.12 Determine the oscillation frequencies of a charged isotropic spatial oscillator in a constant uniform magnetic field. Without a magnetic field the proper oscillation frequency is ω_0. The splitting of oscillation frequencies refers to the *classical Zeeman effect*.

1.5.13 Prove that the *Runge-Lenz vector*

$$\mathcal{A} = Ze^2\frac{\mathbf{r}}{r} - \frac{1}{\mu}[\mathbf{p} \times \mathbf{M}], \qquad \frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}, \qquad \mathbf{M} = [\mathbf{r} \times \mathbf{p}]$$

is conserved for a nonrelativistic electron of charge $-e$ and mass m moving in the field of a nucleus with charge Ze and mass M.

1.5.14 A current of magnitude J flows along an infinitely long straight cylindrical wire of radius r. An electron with the initial velocity $\mathbf{v}_0$ parallel to the wire axis escapes from the wire surface. Find the maximum distance R which the electron can run away from the axis of the wire.

1.5.15 A relativistic particle of mass m and charge e moves in a uniform magnetic field. Determine the variation of the particle energy for one rotation provided the magnetic field varies slowly in time, namely, field variations for the period of motion are small compared with the field magnitude. Prove that the quantity $p_\perp^2/H$ remains constant and represents an *adiabatic invariant*. Calculate the variation of the orbit radius and particle energy if the field changes from magnitude H_1 to H_2.

1.5.16 For the nonrelativistic case, find the equation of motion for the guiding center of a charged particle orbit provided the magnetic field changes slowly at distances of the order of the orbit radius.

1.5.17 At large distances the magnetic field of the Earth looks like a dipole field with magnetic moment $\mu = 8.1 \times 10^{25}\ \text{G} \cdot \text{cm}^3$. The charged particles that come from the solar wind and are trapped within the Earth's magnetic field drift between the poles along the force lines in regions called the *Van Allen radiation belts*.

(a) Assuming that the velocity of a particle at the equator is at an angle α with the equator plane, determine the maximum latitude or polar angle which can be reached by the particle. Find the angle α which allows the particle to reach the Earth's surface if the particle was in the equatorial plane at a distance from the Earth much larger than the radius of the Earth.

(b) Find the period of drift around the Earth for a 10 MeV proton moving in the equatorial plane at a distance of 30 000 km from the center of the Earth.

1.6 Static electromagnetic field

1.6.1 Determine the quadrupole moment of a uniformly charged ellipsoid with respect to its center.

1.6.2 Find the electric field of a uniformly charged ellipsoid at large distances R to an accuracy of fourth order in $1/R$.

1.6.3 Determine the quadrupole moment of a uniformly charged filament of length l if its total charge equals q. Put the coordinate origin at the middle of the filament.

1.6.4 Calculate the quadrupole moment tensor for a neutral system of charges in which two charges $+q$ are located at points $(+a, +a, 0)$ and $(-a, -a, 0)$, and the other two charges $-q$ are placed at points $(+a, -a, 0)$ and $(-a, +a, 0)$. Find the system of coordinates in which this tensor has a diagonal form.

1.6.5 For the system of charges considered in the previous problem, find the energy of the system in an external uniform electric field $\mathbf{E}$.

1.6.6 Two opposite charges of the same magnitude are fixed at a distance l from each other. Find the force and torque acting on the given system if it is placed in the field of a point-like charge at distance $R \gg l$.

1.6.7 Write down the interaction energy $\mathcal{E}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{R})$ between a dipole $\mathbf{d}_1$ at the coordinate origin and a second dipole $\mathbf{d}_2$ at the point $\mathbf{R}$.

1.6.8 Let the dipole $\mathbf{d}_1$ be directed along the z axis. Obtain the expansion of energy $\mathcal{E}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{R})$ in spherical harmonics for fixed values of $\mathbf{d}_1$, $\mathbf{d}_2$, and $|\mathbf{R}|$.

1.6.9 The potential $V(r, \theta)$ for an axially symmetric system of charges at the z $(\theta = 0)$ axis reads

$$V(r, 0) = V_0 \left(1 - \frac{r^2 - a^2}{r\sqrt{r^2 + a^2}} \right), \qquad r > a\ .$$

Find two leading terms of expansion $V(r, \theta)$ at distances $r \gg a$.

1.6.10 A body, confined within a surface similar to a sphere with

$$R(\theta) = R_0[1 + \beta P_2(\cos\theta)],$$

or spheroid, is uniformly charged, $P_2(x)$ being a Legendre polynomial. The total charge equals q. Find the multipole moments of the spheroid in an approximation linear in β.

1.6.11 Show that the uniform magnetic field $\mathbf{H}$ directed along the z axis can be described by the vector potential $\mathbf{A}_1 = (0, Hx, 0)$.
Transform to the potential $\mathbf{A}_2 = [\mathbf{H} \times \mathbf{r}]/2$ using a gauge transformation.

1.6.12 A magnetic field has a component in the z direction and decreases in the same direction with the constant gradient $\partial H_z/\partial z = -h = const$. Could this field be parallel to the z axis in the whole space? Find the radial components of the field outside the z axis. Plot the lines of force.

1.6.13 Derive the formula $\mathbf{F} = (\boldsymbol{\mu}\boldsymbol{\nabla})\mathbf{H}$ for the force acting on a magnetic dipole in an inhomogeneous field.

1.6.14 Find the equation of the line of force of a magnetic dipole in polar coordinates. Determine how the field changes along the line of force.

1.6.15 Determine the energy of interaction between a magnetic dipole μ moving with velocity $v \ll c$ and a fixed nucleus of charge Ze.

1.7 Free electromagnetic field

1.7.1 What conditions should be satisfied for the complex amplitude of a plane monochromatic wave, $A_0^\mu = (A_0^0, \mathbf{A}_0)$, and the wave 4-vector $k^\mu = (\omega/c, \mathbf{k})$?

1.7.2 Find the energy-momentum tensor of a linearly polarized monochromatic *traveling* plane wave.

1.7.3 Find the energy-momentum tensor of a linearly polarized monochromatic *standing* plane wave.

1.7.4 Show that the scalar potential of an arbitrary *free* electromagnetic field can be reduced to zero by using the residual gauge transformation, without violating the Lorentz gauge.

1.7.5 Write the Maxwell equations for the Fourier amplitudes obtained by expanding an arbitrary plane wave in terms of monochromatic plane waves.

1.7.6 Find the expression for the Fourier amplitudes of a monochromatic plane wave.

1.8 The retarded potentials and radiation

1.8.1 Determine the electric and magnetic field of a harmonically oscillating dipole at distances r much larger than the dipole size a but, maybe, comparable with the wavelength λ.

1.8.2 Prove that, for a closed system of charged particles with the same charge-to-mass ratio, there is neither electric nor magnetic dipole radiation.

1.8.3 In the *Rutherford atomic model* two opposite charges (e_1, m_1) and (e_2, m_2) orbit around each other along a circular trajectory of radius R under the action of the Coulomb attraction. Determine the energy loss due to radiation per revolution. Find the dependence of the distance between charges on time. Determine the time at which one charge will fall on the other.

1.8.4 Let us consider the classical model of the helium atom in which two electrons with charges $-e$, located at diametrically opposite points, orbit the infinitely heavy nucleus of charge $+2e$. What is the intensity of the emitted electric and magnetic dipole radiation? What is the intensity of quadrupole radiation? How does the orbit radius change in time?

1.8.5 Two like charges (e_1, m_1) and (e_2, m_2) are subjected to a head-on collision. Determine the total radiated energy if the relative velocity at infinity is given by $v_\infty \ll c$. Consider the dipole $e_1/m_1 \neq e_2/m_2$ and quadrupole $e_1/m_1 = e_2/m_2$ cases.

1.8.6 A body, confined within a surface similar to a sphere with

$$R(\theta) = R_0[1 + \beta P_2(\cos\theta)],$$

i.e. a spheroid, is uniformly charged, $P_2(x)$ being a Legendre polynomial. The total charge equals q. The small parameter $\beta = \beta(t) \ll 1$ oscillates in time with frequency ω. Keeping lowest terms of expansion in β, calculate the angular distribution and total power of radiation in the long wave approximation.

1.8.7 A charge e with mass m is placed at a weightless rod of length l.
(1) Find the temporal variation law for the angular velocity $\omega(t)$ if $\omega(0) = \omega_0$ at the initial time moment.
(2) Determine the average angular distribution $\overline{dI/d\Omega}$ and the total intensity $\overline{I}$ of the radiation.

1.8.8 An electron travels around a circle in the uniform magnetic field $\mathbf{H}$. Find the temporal behavior of the electron kinetic energy $\mathcal{E}(t)$. Determine the electron trajectory in the nonrelativistic limit $v \ll c$.

1.8.9 Determine the average angular distribution $\overline{dI/d\Omega}$ and the total radiation intensity $\overline{I}$ for two like charges e traveling uniformly and nonrelativistically around a circle of radius a with angular velocity ω in diametrically opposite positions.

1.8.10 Two like charges e travel uniformly and nonrelativistically with angular velocity ω along a circle of radius a. How should the arrangement of charges differ from diametrically opposite positions in order to have equal intensities of electric dipole and quadrupole radiation?

1.8.11 A point dipole $\mathbf{d}$ travels along a circle of radius R with angular velocity ω. The dipole is oriented along the radius-vector $\mathbf{d}||\mathbf{R}$. Find the intensities of the electric dipole, quadrupole and magnetic dipole radiation for $R \ll \lambda = 2\pi c/\omega$.

1.8.12 Two like dipoles of magnitude $\mathbf{d}$ are located at distance R from the center of a weightless rod and oppositely oriented along the rod. Determine the temporal behavior of the angular velocity $\omega(t)$ if at the initial time moment $\omega(0) = \omega_0$. The mass of the dipole is m and $R \ll c/\omega$.

1.8.13 The velocity of a charged particle decreases from v_0 to zero during the time τ. Find the angular distribution of the bremsstrahlung (braking radiation), assuming the acceleration to be constant. What radiation pulse duration Δt will be registered by the receiver at rest if it is set at angle θ with respect to the direction of the particle motion?

1.8.14 An electron is subjected to a head-on collision with a fixed (infinitely heavy) point dipole $\mathbf{d}$. Determine the energy loss due to radiation as a result of this collision. The electron moves in a straight line.

1.8.15 A uniformly charged rod with the linear charge density ρ and length l rotates around its end. Find the temporal behavior of its angular velocity $\omega(t)$ if $\omega(0) = \omega_0$. The rod mass is m.

1.8.16 A uniformly charged rod with the linear charge density ρ and length $2l$ rotates around an axis running through the middle of the rod. Find the temporal behavior of the angular velocity $\omega(t)$ if $\omega(0) = \omega_0$. The rod mass is $2m$.

1.8.17 Consider the neutral system of four charged particles fixed at the corners of a $2a \times 2a$-sized square, two positive charges each $+q$ being at one diagonal and two negative charges each $-q$ being at the other diagonal. The square wholly lies in the xy plane and the z axis runs through the center of the square. The square rotates around the z axis with some angular velocity ω. Find the temporal behavior of the angular velocity $\omega(t)$ if $\omega(0) = \omega_0$. The masses of the particles are the same and equal to m.

1.8.18 At the initial time a magnetic moment $\boldsymbol{\mu}$ is set in the direction normal to an external uniform magnetic field $\mathbf{H} = (0, 0, H)$. Then, the magnetic moment starts to precess around the magnetic field direction. Find $\mu_z(t)$ if the gyromagnetic ratio equals γ.

1.8.19 An oscillatory circuit contains a capacitor and an inductance coil. The natural oscillation frequency is ω_0. Let the coil be ideal, i.e. dissipationless, and the capacitor consist of two parallel plates of area s, the distance between the plates being a. The size of the capacitor is small compared with $\lambda = 2\pi c/\omega$. Determine the Q-factor of the circuit, i.e. 2π multiplied by the ratio of stored energy to energy losses.

1.8.20 An oscillatory circuit contains a capacitor and an inductance coil. The natural oscillation frequency is ω_0. The capacitor is ideal or dissipationless. The coreless coil consists of N turns, each of the turns having an area s. The coil length is a. The coil size is small compared with $\lambda = 2\pi c/\omega$. Determine the circuit Q-factor, i.e. 2π multiplied by the ratio of stored energy to energy losses.

1.8.21 A nonrelativistic charged particle with charge q and mass m slides frictionlessly on the surface of revolution in the horizontal plane. The axis of revolution is vertical. The gravitational force $m\mathbf{g}$ and the reaction force normal to the contact surface act on the particle. The particle dissipates its energy due to electromagnetic radiation and, for one rotation, the energy losses are small. Determine the time dependence $z(t)$ for the following surfaces of revolution:
(1) surface of a cone $z = kr$,
(2) surface of a paraboloid $z = r^2/a$.

1.8.22 A harmonically oscillating dipole is set at height h above an ideally conducting metallic plane. For the cases $h \ll \lambda$ and $h \gg \lambda$, find the dipole radiation intensity as a function of the observation angle and the angle between the dipole and the normal to the metallic plane.

1.9 Electromagnetic field of relativistic particles

1.9.1 Determine the electric and magnetic fields of an arbitrarily moving charge.

1.9.2 Determine the electromagnetic field induced by an electron moving uniformly at velocity $\mathbf{v}$.

1.9.3 Determine the interaction force $\mathbf{F}$ of two electrons moving parallel to each other with the same velocities $\mathbf{V}$.

1.9.4 Find the instantaneous angular distribution of radiation power $dW/d\Omega$ for a relativistic charged particle in two cases:
(1) particle velocity $\mathbf{v}$ and acceleration $\mathbf{w}$ are parallel at the retarded time t';
(2) velocity and acceleration are mutually perpendicular at the retarded time t'.
Analyze the nonrelativistic $v/c \ll 1$ and ultrarelativistic $v/c \sim 1$ limits.

1.9.5 Find the radiation energy of a relativistic electron in a uniform magnetic field for one rotation. Find the total power (in Megawatt) of *synchrotron radiation* in a collider of electrons and positrons with the energy 100 GeV. The circumference of the collider is 30 km and the number of accelerated particles in the ring is 5×10^{12}. Estimate the typical wavelength of radiation.

1.9.6 A beam of relativistic electrons traverses a plane capacitor (*electric undulator*) under alternating voltage of frequency ω_0. Find the frequency of *undulator radiation* as a function of the angle θ between an observer and the direction of beam motion.

1.9.7 Determine the temporal behavior for the energy of a relativistic electron in a uniform magnetic field, the electron velocity being perpendicular to the field. Verify that in the $t \to \infty$ limit, the behavior agrees with the result for nonrelativistic electrons.

1.9.8 Assuming that a particle in a linear accelerator travels along a straight line under the action of a constant force, determine the energy losses due to radiation. Find the energy conversion efficiency for an ultrarelativistic particle.

1.10 The scattering of electromagnetic waves

1.10.1 A linearly polarized monochromatic plane wave is scattered by a free electron. Determine the polarization of the scattered electromagnetic field.

1.10.2 (1) Find the differential and total cross sections for the scattering of linearly polarized and natural (unpolarized) light by a damped oscillator. The damping is due to viscous friction.
(2) Find the total cross section of *absorption*.

1.10.3 Determine the differential cross section $d\sigma/d\Omega$ for the scattering of a circularly polarized monochromatic electromagnetic plane wave by an electron in the strong uniform magnetic field $\mathbf{H}_0$, taking into account radiation damping. Consider the case $\mathbf{k}||\mathbf{H}_0$ where $\mathbf{k}$ is the wave vector of the incident wave.

1.10.4 Determine the differential cross section $d\sigma/d\Omega$ for the scattering of an unpolarized monochromatic electromagnetic plane wave by a charged harmonic oscillator in the strong uniform magnetic field $\mathbf{H}_0$, taking into account radiation damping. Consider the case $\mathbf{k}||\mathbf{H}_0$.

1.10.5 Find the differential cross section for the scattering of a circularly polarized monochromatic plane electromagnetic wave of frequency ω and wave vector $\mathbf{k}$ at a system having the magnetic moment $\boldsymbol{\mu}$ and gyromagnetic ratio γ_e, provided that the magnetic moment $\boldsymbol{\mu}$ experiences small oscillations around the mean position $\boldsymbol{\mu}_0$ under the influence of the incident wave. Consider the case of a small gyromagnetic ratio. What is the criterion of smallness in the problem?

1.10.6 Find the differential and total cross sections for the scattering of a linearly polarized monochromatic electromagnetic plane wave of frequency ω on a capacitor. The capacitor is composed of two parallel plates of area s located at a distance a from each other. The plates are short-circuited. The electric field of the incident wave is perpendicular to the plates. The size of the capacitor is small compared with the wavelength.

1.10.7 Find the differential and total cross sections for the scattering of a linearly polarized monochromatic electromagnetic plane wave of frequency ω on a coil. The coil consists of N turns, each of the turns having an area s. The coil length is a. The coil is short-circuited. The coil size is small compared with the wavelength.

1.10.8 Estimate the density of matter in the *solar corona* if the corona brightness, observed in the course of a solar eclipse, is smaller by a factor 10^6 in comparison with the brightness of the *solar photosphere* or visible surface of the Sun.

1.10.9 For relatively young stars, e.g. *novae*, the luminosity, i.e. total energy radiated per unit time, varies drastically for several days. In the beginning the luminosity grows abruptly and then decreases slowly to the initial magnitude. The mechanism of the phenomenon is a detachment of the gas shell of star due to disturbances of the equilibrium state. In essence, the light pressure on the gas shell or, more precisely, plasma shell, three fourths of which are ions of hydrogen, becomes larger than the gravitational attraction. Estimate the *critical luminosity* or *Eddington limit* for a star of mass M and radius R.

1.10.10 Find the differential and total cross section for the scattering of a linearly polarized monochromatic electromagnetic wave on a superconducting sphere of radius $r \ll \lambda$. The fields of the charges and currents, induced by the external electromagnetic wave field at the sphere surface, compensate completely the external fields inside the sphere.