CONTENTS

PREFACE

CHAPTER 1 - GENERALITIES ON ELLIPTIC VARIATIONAL INEQUALITIES AND ON THEIR APPROXIMATIONS

1. Introduction 1
2. Functional Context 1
   2.1. Notations 1
   2.2. EVI of first kind 2
   2.3. EVI of second kind 2
   2.4. Remarks 2
3. Existence and Uniqueness results for EVI of first kind 3
   3.1. A Theorem of Existence and Uniqueness 3
   3.2. Remarks 5
4. Existence and Uniqueness results for EVI of second kind 7
5. Internal Approximation of EVI of first kind 11
   5.1. Introduction 11
   5.2. The Continuous problem 12
   5.3. The Approximate problem 12
      5.3.1. Approximation of V and K 12
      5.3.2. Approximation of (P_1) 12
   5.4. Convergence results 13
6. Internal Approximation of EVI of second kind 16
   6.1. The Continuous Problem 16
   6.2. Definition of the approximate problem 16
      6.2.1. Approximation of V 17
      6.2.2. Approximation of $j(\cdot)$ 17
      6.2.3. Approximation of (P_2) 18
   6.3. Convergence results 18
7. References 21

CHAPTER 2 - APPLICATION OF THE FINITE ELEMENT METHOD TO THE APPROXIMATION OF SOME SECOND ORDER EVI

1. Introduction 22
2. An example of EVI of the first kind : the obstacle problem 22
   2.1. The continuous problem 23
   2.2. Existence and Uniqueness results 24
   2.3. Interpretation of (2.1) as a free boundary problem 26
   2.4. Regularity of the solutions 26
   2.5. Finite Element Approximation of (2.1) 29
      2.5.1. Approximation of V and K 30
      2.5.2. The approximate problems 31
   2.6. Convergence results 31
   2.7. Comments on the error estimates 39
      2.7.1. Piecewise linear approximation 39
      2.7.2. Piecewise quadratic approximation 39
   2.8. Iterative solution of the approximate problem 39

3. A second example of EVI of the first kind : the elasto-plastic torsion problem 41
3.1. Formulation. Preliminary results 41
3.2. Physical motivation 42
3.3. Regularity properties and exact solutions 44
3.3.1. Regularity results 44
3.3.2. Exact solutions 45
3.4. An equivalent variational formulation 46
3.5. Finite Element Approximation of (3.1) 47
3.5.1. Approximation of V and K 47
3.5.2. The approximate problem 48
3.5.3. Remarks on the use of higher order finite elements 48
3.6. Convergence results. General case 49
3.6.1. A density lemma 49
3.6.2. A convergence theorem 50
3.7. Error estimates 52
3.7.1. One-dimensional case 52
3.7.2. Two-dimensional case 55
3.8. A dual iterative method for solving (3.1) and (3.2) 58
3.8.1. The continuous case 58
3.8.2. The discrete case 59

4. A third example of EVI of the first kind : a simplified Signorini problem 61
4.1. The continuous problem : existence and uniqueness results 61
4.2. Regularity of the solution 63
4.3. Interpretation of (4.5) as a free boundary problem 63
4.4. Finite element approximation of (4.5) 66
4.4.1. Approximation of V and K 66
4.4.2. The approximate problems 66
4.5. Convergence results. (General case) 67
4.5.1. A density lemma 67
4.5.2. Convergence theorem 68
4.6. Iterative methods for solving the discrete problem 71
4.6.1. Solution by an over-relaxation method 71
4.6.2. Solution by a duality method 71

5. An example of EVI of the second kind : A simplified friction problem 78
5.1. The continuous problem. Existence and Uniqueness results 78
5.2. Regularity of the solution 79
5.3. Existence of a multiplier 79
5.4. Finite element approximation of (5.5) 82
5.4.1. Approximation of V 82
5.4.2. Approximation of $j(\cdot)$ 82
5.4.3. The approximate problem 83
5.5. Convergence results 83
5.6. Iterative methods for solving (P^k_{2h}) 88
5.6.1. Solution of (P^k_{2h}) by relaxation methods 88
5.6.2. Solution of (P^k_{2h}) by duality methods 89

6. A second example of EVI of the second kind : The flow of a viscous, plastic fluid in a pipe 90
6.1. The continuous problem. Existence and Uniqueness results 90
6.2. Physical motivation 91
6.3. Regularity properties 92
6.4. Further properties 92

6.5. Exact solutions 93
6.5.1. Example 1 93
6.5.2. Example 2 94
6.6. Existence of multipliers 94
6.7. Finite element approximation of (6.1) 98
6.7.1. Definition of the approximate problem 98
6.7.2. Convergence of the approximate solutions (General case) 99
6.7.3. Convergence of the approximate solutions ($f \in L^2(\Omega)$) 101
6.8. The case of a circular domain with f = constant 103
6.8.1. Exact solutions and regularity properties 103
6.8.2. Approximation by finite element of order 1 103
6.8.3. Error estimates 104
6.8.4. Generalization 110
6.9. Iterative solution of the continuous and approximate problems by Uzawa's algorithm 110

CHAPTER 3 - ON THE APPROXIMATION OF PARABOLIC VARIATIONAL INEQUALITIES. 116

1. Introduction. References. 116
2. Formulation and statements of the main results 116
3. Numerical schemes for parabolic linear equations 117
4. Approximation of PVI of the first kind 120
5. Approximation of PVI of the second kind 122
6. Application to a specific example : time dependent flow of a Bingham fluid in a cylindrical pipe 123
6.1. Formulation of the problem. Existence and uniqueness theorem 123
6.2. The asymptotic behaviour of the continuous solution 124
6.3. On the asymptotic behaviour of the discrete solution 126
6.4. Remarks 129

APTER 4 - APPLICATIONS OF ELLIPTIC VARIATIONAL INEQUALITY METHODS TO THE SOLUTION OF SAME NONLINEAR ELLIPTIC EQUATIONS 130

1. Introduction 130
2. Theoretical and Numerical Analysis of some mildly Nonlinear Elliptic equations 130
2.1. Formulation of the continuous problem 130
2.2. A variational inequality related to (P) 132
2.2.1. Definition of the variational inequality 132
2.2.2. Properties of $j(\cdot)$ 132
2.2.3. Existence and uniqueness results for (π) 134
2.3. Equivalence between (P) and (π) 134
2.4. Some comments on the continuous problem 146
2.5. Finite element approximation of (π) and (P) 147
2.5.1. Definition of the approximate problem 147
2.5.2. Convergence of the approximate solutions 149
2.6. Iterative methods for solving the discrete problem 155
2.6.1. Introduction 155
2.6.2. Formulation of the discrete problem 155

2.6.3. Gradient methods 156
2.6.4. Newton's method 157
2.6.5. Relaxation and over-relaxation methods 158
2.6.6. Alternating Direction methods 160
2.6.7. Conjugate Gradient methods 161
2.6.8. Comments 162

3. A subsonic flow problem 163
3.1. Formulation of the continuous problem 163
3.2. Variational formulation of subsonic problems 164
3.3. Existence and Uniqueness properties for the problem (3.8) 167
3.4. Comments 170

CHAPTER 5 - DECOMPOSITION-COORDINATION METHODS BY AUGMENTED LAGRANGIANS. APPLICATIONS 171

1. Introduction 171
1.1. Motivation 171
1.2. Principle of the methods 173

2. Properties of (P) and of the saddle-points of $\mathcal{L}$ and $\mathcal{L}_r$ 173
2.1. Existence and uniqueness properties for (P) 173
2.2. Properties of the saddle-points of $\mathcal{L}$ and $\mathcal{L}_r$ 174

3. Description of the algorithms 176
3.1. First algorithm 177
3.2. Second algorithm 177

4. Convergence of ALG 1 178
4.1. General Case 178
4.2. Finite dimensional case 183
4.3. Comments on the use of ALG 1. Further remarks 186

5. Convergence of ALG 2 189
5.1. Orientation 189
5.2. General case 189
5.3. Finite dimensional case 194
5.4. Comments on the choice of ρ and r 194
5.4.1. Some remarks 194
5.4.2. On the choice of ρ and r 194

6. Applications 195
6.1. Bingham flow in a cylindrical pipe 195
6.2. Elastic-plastic torsion of a cylindrical bar 197
6.3. A nonlinear Dirichlet problem 198
6.4. Application to the solution of mildly nonlinear systems 201
6.5. Solution of Elliptic Variational Inequalities on intersections of convex sets 205
6.5.1. Formulation of the problem 205
6.5.2. Decomposition of (6.61),(6.62) 206
6.5.3. Solution of (6.62) by ALG 1 207
6.5.4. Solution of (6.62) by ALG 2 208

7. General comments 209

CHAPTER VI - ON THE COMPUTATION OF TRANSONIC FLOWS 210

1. Introduction 210

2. Generalities 210

3. Mathematical model for the transonic flow problem 212
   3.1. Basic assumptions and generalities 212
   3.2. Equations of the flow 212
   3.3. Formulation of the entropy condition 213

4. Reduction to an optimal control problem 214

5. Approximation 216
   5.1. Generalities 216
   5.2. Approximation of the state equation and of the cost function 218
   5.3. Approximation of the entropy condition 219
      5.3.1. A regularization method 219
      5.3.2. A method using (3.7) 221
   5.4. Approximation of X 224

6. Iterative solution of the approximate problems 224
   6.1. Preliminary statements, generalities 224
   6.2. A saddle-point formulation of the approximate problem. Augmented lagrangian 224
   6.3. Iterative solution of the approximate problem via $\mathcal{L}_r$ 226
      6.3.1. Description of the algorithm 226
      6.3.2. Solution of (6.10) 226
      6.3.3. Solution of (6.8) 226
      6.3.4. Computation of $\frac{\partial \mathcal{L}_r}{\partial \xi_h}$ 227
   6.4. Computational considerations 228

7. A numerical experiment 228

8. Comments. Conclusion 230

References 231