

Contents

Preface	v
I Feynman–Kac-type theorems and Gibbs measures	1
1 Heuristics and history	3
1.1 Feynman path integrals and Feynman–Kac formulae	3
1.2 Plan and scope	7
2 Probabilistic preliminaries	11
2.1 An invitation to Brownian motion	11
2.2 Martingale and Markov properties	21
2.2.1 Martingale property	21
2.2.2 Markov property	25
2.2.3 Feller transition kernels and generators	29
2.2.4 Conditional Wiener measure	32
2.3 Basics of stochastic calculus	33
2.3.1 The classical integral and its extensions	33
2.3.2 Stochastic integrals	34
2.3.3 Itô formula	42
2.3.4 Stochastic differential equations and diffusions	46
2.3.5 Girsanov theorem and Cameron–Martin formula	50
2.4 Lévy processes	53
2.4.1 Lévy process and Lévy–Khintchine formula	53
2.4.2 Markov property of Lévy processes	57
2.4.3 Random measures and Lévy–Itô decomposition	61
2.4.4 Itô formula for semimartingales	64
2.4.5 Subordinators	67
2.4.6 Bernstein functions	69
3 Feynman–Kac formulae	71
3.1 Schrödinger semigroups	71
3.1.1 Schrödinger equation and path integral solutions	71
3.1.2 Linear operators and their spectra	72
3.1.3 Spectral resolution	78
3.1.4 Compact operators	80

3.1.5	Schrödinger operators	81
3.1.6	Schrödinger operators by quadratic forms	85
3.1.7	Confining potential and decaying potential	87
3.1.8	Strongly continuous operator semigroups	89
3.2	Feynman–Kac formula for external potentials	93
3.2.1	Bounded smooth external potentials	93
3.2.2	Derivation through the Trotter product formula	95
3.3	Feynman–Kac formula for Kato-class potentials	97
3.3.1	Kato-class potentials	97
3.3.2	Feynman–Kac formula for Kato-decomposable potentials	108
3.4	Properties of Schrödinger operators and semigroups	112
3.4.1	Kernel of the Schrödinger semigroup	112
3.4.2	Number of eigenfunctions with negative eigenvalues	113
3.4.3	Positivity improving and uniqueness of ground state	120
3.4.4	Degenerate ground state and Klauder phenomenon	124
3.4.5	Exponential decay of the eigenfunctions	126
3.5	Feynman–Kac–Itô formula for magnetic field	131
3.5.1	Feynman–Kac–Itô formula	131
3.5.2	Alternate proof of the Feynman–Kac–Itô formula	135
3.5.3	Extension to singular external potentials and vector potentials	138
3.5.4	Kato-class potentials and L^p – L^q boundedness	142
3.6	Feynman–Kac formula for relativistic Schrödinger operators	143
3.6.1	Relativistic Schrödinger operator	143
3.6.2	Relativistic Kato-class potentials and L^p – L^q boundedness	149
3.7	Feynman–Kac formula for Schrödinger operator with spin	150
3.7.1	Schrödinger operator with spin	150
3.7.2	A jump process	152
3.7.3	Feynman–Kac formula for the jump process	154
3.7.4	Extension to singular potentials and vector potentials	157
3.8	Feynman–Kac formula for relativistic Schrödinger operator with spin	162
3.9	Feynman–Kac formula for unbounded semigroups and Stark effect	166
3.10	Ground state transform and related diffusions	170
3.10.1	Ground state transform and the intrinsic semigroup	170
3.10.2	Feynman–Kac formula for $P(\phi)_1$ -processes	174
3.10.3	Dirichlet principle	181
3.10.4	Mehler’s formula	184
4	Gibbs measures associated with Feynman–Kac semigroups	190
4.1	Gibbs measures on path space	190
4.1.1	From Feynman–Kac formulae to Gibbs measures	190
4.1.2	Definitions and basic facts	194

4.2	Existence and uniqueness by direct methods	201
4.2.1	External potentials: existence	201
4.2.2	Uniqueness	204
4.2.3	Gibbs measure for pair interaction potentials	208
4.3	Existence and properties by cluster expansion	217
4.3.1	Cluster representation	217
4.3.2	Basic estimates and convergence of cluster expansion	223
4.3.3	Further properties of the Gibbs measure	224
4.4	Gibbs measures with no external potential	226
4.4.1	Gibbs measure	226
4.4.2	Diffusive behaviour	238

II Rigorous quantum field theory 245

5	Free Euclidean quantum field and Ornstein–Uhlenbeck processes	247
5.1	Background	247
5.2	Boson Fock space	249
5.2.1	Second quantization	249
5.2.2	Segal fields	255
5.2.3	Wick product	257
5.3	$\mathcal{Q}$ -spaces	258
5.3.1	Gaussian random processes	258
5.3.2	Wiener–Itô–Segal isomorphism	260
5.3.3	Lorentz covariant quantum fields	262
5.4	Existence of $\mathcal{Q}$ -spaces	263
5.4.1	Countable product spaces	263
5.4.2	Bochner theorem and Minlos theorem	264
5.5	Functional integration representation of Euclidean quantum fields	268
5.5.1	Basic results in Euclidean quantum field theory	268
5.5.2	Markov property of projections	271
5.5.3	Feynman–Kac–Nelson formula	274
5.6	Infinite dimensional Ornstein–Uhlenbeck process	276
5.6.1	Abstract theory of measures on Hilbert spaces	276
5.6.2	Fock space as a function space	279
5.6.3	Infinite dimensional Ornstein–Uhlenbeck-process	282
5.6.4	Markov property	288
5.6.5	Regular conditional Gaussian probability measures	290
5.6.6	Feynman–Kac–Nelson formula by path measures	292
6	The Nelson model by path measures	293
6.1	Preliminaries	293

6.2	The Nelson model in Fock space	294
6.2.1	Definition	294
6.2.2	Infrared and ultraviolet divergences	296
6.2.3	Embedded eigenvalues	298
6.3	The Nelson model in function space	298
6.4	Existence and uniqueness of the ground state	303
6.5	Ground state expectations	309
6.5.1	General theorems	309
6.5.2	Spatial decay of the ground state	315
6.5.3	Ground state expectation for second quantized operators . . .	316
6.5.4	Ground state expectation for field operators	322
6.6	The translation invariant Nelson model	324
6.7	Infrared divergence	328
6.8	Ultraviolet divergence	333
6.8.1	Energy renormalization	333
6.8.2	Regularized interaction	335
6.8.3	Removal of the ultraviolet cutoff	339
6.8.4	Weak coupling limit and removal of ultraviolet cutoff	344
7	The Pauli–Fierz model by path measures	351
7.1	Preliminaries	351
7.1.1	Introduction	351
7.1.2	Lagrangian QED	352
7.1.3	Classical variant of non-relativistic QED	356
7.2	The Pauli–Fierz model in non-relativistic QED	359
7.2.1	The Pauli–Fierz model in Fock space	359
7.2.2	The Pauli–Fierz model in function space	363
7.2.3	Markov property	369
7.3	Functional integral representation for the Pauli–Fierz Hamiltonian . .	372
7.3.1	Hilbert space-valued stochastic integrals	372
7.3.2	Functional integral representation	375
7.3.3	Extension to general external potential	381
7.4	Applications of functional integral representations	382
7.4.1	Self-adjointness of the Pauli–Fierz Hamiltonian	382
7.4.2	Positivity improving and uniqueness of the ground state . . .	392
7.4.3	Spatial decay of the ground state	398
7.5	The Pauli–Fierz model with Kato class potential	399
7.6	Translation invariant Pauli–Fierz model	401
7.7	Path measure associated with the ground state	408
7.7.1	Path measures with double stochastic integrals	408
7.7.2	Expression in terms of iterated stochastic integrals	412
7.7.3	Weak convergence of path measures	415

7.8	Relativistic Pauli–Fierz model	418
7.8.1	Definition	418
7.8.2	Functional integral representation	420
7.8.3	Translation invariant case	423
7.9	The Pauli–Fierz model with spin	424
7.9.1	Definition	424
7.9.2	Symmetry and polarization	427
7.9.3	Functional integral representation	434
7.9.4	Spin-boson model	447
7.9.5	Translation invariant case	448
8	Notes and References	455
	Bibliography	473
	Index	499