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The Equations of Fluid Motion

Upon the word, accoutered as I was, I plunged
in and bade him follow. So indeed he did.

Shakespeare, *Julius Caesar*

We live in an incredibly dynamical universe. Any chance observation bears witness to the enormous diversity of motions and interactions that dominate the structure of matter in astrophysical environments. It is the aim of this book to put some of this in context. We see that gravitation, the ultimate structuring force on the scales typical in astrophysical problems, plays a key role in understanding how matter evolves. But we also must reckon with magnetic fields, turbulence, and many of the same processes that dominate laboratory studies.

1.1
Introductory Remarks

When you think of a fluid, the idea of a structureless deformable *continuous* medium comes to mind. Other properties, ingrained in childhood, are also likely to occur to you; things like incompressibility, viscosity (if you were really precocious), and perhaps even the statement that a fluid will not support shear and will yield freely in the presence of an applied force. So what have these to do with an astrophysical context? More than these definitions and properties might lead you to suspect. Of course, you will likely think, since stars are gaseous (something we have known since the 1860s), that we should be dealing with kinetic theory and gas dynamics. The processes of rarefied media are likely to dominate our understanding of cosmic bodies. This is not quite true. Since a star is composed of gas which is homogeneous (in *most* cases) and which acts collectively to create its own gravitational field, it mimics rather well the behavior of a fluid moving (or sitting) under gravity. The collision times are so short (or, put another way, *the mean free paths are so short compared with any scale lengths in the medium*) in the interior of the star that any disturbances can be washed out and the structure can be described as contin-

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uous. Naturally, this is, for the moment, only an assertion. We shall prove it in due time.

The most crucial point is that stars and all other cosmic matter can be treated as an ensemble object or system *when we have carefully chosen some scales of length and time*. In a nutshell, the reason for this book is that we can very often, at some magnification of scale or some rate of clock ticking, apply a fluid approximation to the problems at hand. This book is meant to provide the machinery, both computational and conceptual, with which to begin treating dynamical and static problems posed by *fluids* in a nonterrestrial environment.

1.2 Equations of Motion

1.2.1 Distribution Functions

We start with a homogeneous medium of identical particles, forgetting for the moment that this may be *too* restrictive an assumption. Imagine that this group of particles is characterized by a *global* velocity distribution. Also, assume that we can know this distribution function and that the positional information can eventually be derived for the particles as well. Let us start with a gas that consists of a collection of myriads of particles, all of identical mass. If we assume that these particles execute collective motions, we will be able to take ensemble averages and treat them as if they were a continuous medium. This is what we mean by a fluid in an astrophysical context. But before we can reach the stage of describing the matter as a classical substance, we need to consider the microscale phenomena and how to incorporate them into a macroscopic description of the motion and thermal properties. To do this, we begin with a statistical mechanics treatment and then generalize from there.

Let us say that there exists a possibly time-dependent distribution function f, which is a function of $\mathbf{x}$, the particle positions, and $\mathbf{v}$, their velocities, and which provides a complete description of the chance of any single particle having a specific position and velocity at any time. We assume that the particle motions are individually governed by any forces imposed externally by the medium and also by any mutual interactions (i.e., collisions). By this we mean that the particles "see" each another through short-range interactions and also collectively through bulk or ensemble interactions. A good example of the former is the electrostatic interaction between charged particles in a plasma, while the latter is exemplified by the integrated gravitational field resulting from the distribution of the whole mass of the material. Both feed back into the distribution function, both alter the microscale properties, and therefore both internal and external forces must be considered if we are to calculate the physical attributes of the medium in the large.

Some constraints can be placed on the form of the distribution function right from the start. For one thing, it should depend only on the magnitude of the velocity (momentum), not on its direction. Another way of saying this is that it should be symmetric with respect to the spatial and velocity components, that is, $f(\mathbf{x}, \mathbf{v}) = f(-\mathbf{x}, -\mathbf{v})$. Since the distribution function is assumed to be a measure of the probability that a particle will have a specific position and velocity, f must be integrable and normalizable. It need not be algebraic; for example, a delta function, $\delta(\mathbf{x})\delta(\mathbf{v})$, is allowable. Now the hyperspace we are dealing with is well known. It is the *phase space* of the ensemble, that collection of individual momenta and positions familiar from classical mechanics. We can picture this collection as a group of free particles all passing through a box in which we have placed an observer. This observer has no idea where these came from or where they are headed, but can at least describe them in the vicinity. They will arrive in this corner of the world, interact (perhaps), and then exit. The overall result is that a complete distribution function can be specified and, if this observer is not too egocentric, this function can even be generalized to describe all of spacetime containing these particles.

A few things are then clear. The distribution function must be scalar and depend only on scalar quantities. That is, it cannot depend on the placement of the observer within the ensemble of particles. It must depend only on scalar quantities, although these may themselves be combinations of many physical properties. If the distribution function is to be global, it must be characterized by some global parameter which is a constant for the system under study. The assertion that the distribution does not change an inversion suggests that it cannot be a pseudoscalar. So f must be positive everywhere. Since the fact that the distribution function is defined in terms of a probability, we would not know how to interpret *negative* values for f. But this property of probabilities is very important for our considerations that follow.

If every particle in a gas has a position and a velocity, we might ask what the mean value is of any quantity connected with this distribution. For instance, we may wish to know the average velocity, or what the average distance is that a particle may be away from the statistical center of the distribution. These are *moments* of the ensemble. Even though we cannot observe the motion of every constituent component of a body, and cannot distinguish the histories of the individual particle trajectories, we can still say something about the most probable values that any of the measurable quantities will have.

Let us examine this physical picture in terms of simple probabilities, reducing the distribution function to only one independent quantity. We must be careful to choose a physically meaningful attribute. For instance, position means something in an extended body. But color probably does not. Even if the particles have different colors, masses, or whatever, we can always ignore these attributes until some quantity that we happen to be interested in requires including them. Take, for example, the position of a particle. If the probability of being some distance from a fixed point in space, $x_0 = 0$, is $P(x)$, then the mean value for the displacement is $\langle x \rangle = \int xP(x)dx$. Now assume that we have a one-dimensional distribution, but one that extends over the range $(-\infty, \infty)$. Since the probability of being on the negative side or the positive side of the reference point is assumed to be the same, the mean value for the position must vanish; that is, on average the particle will be at the reference point. Another way of saying this is that the integrand consists of a symmetric and an antisymmetric part and therefore vanishes over the whole space. But this clearly does not make sense if the ensemble is extended. There must be some other way of treating the fact that many of the particles, although perhaps equally distributed on the two axes, may not be concentrated at the nominal zero point. We require a quantity that does not vanish on integrating over the whole ensemble, $\langle x^2 \rangle$. This is a measure of the dispersion of the particles in space, and unlike $\langle x \rangle$, $\langle x^2 \rangle$ is finite. Now we have both a symmetric function and a symmetric interval and the mean value therefore does not vanish.

This has been a rather long digression. It is, however, prompted by the need to place the process of taking moments in context.

1.2.2
Moments of the Distribution Function

Of all the quantities that you can think of as characterizing this gas, the most obvious ones are functions of velocity and density. This is just a product of our Newtonian bias. We will separate the equations for the velocity into two components. One is the mean velocity, which we shall write as V_i, and the other is the random motion, which is assumed to have a mean of zero. This velocity we shall call u_i. All of the moments will be taken assuming that the distribution function is taken over the random velocities *only*. For instance, there are quantities around which classical descriptions in physics revolve: the *number density*, $n(\mathbf{x})$, the *momentum flux*, $n(\mathbf{x})\mathbf{V}(\mathbf{x})$, and the *energy density*, $\frac{1}{2}n(\mathbf{x})\mathbf{V} \cdot \mathbf{V}$. You will notice that each of these is a function of some power of the velocity although each depends only on space and time, not on the internal velocity distribution of the particles.

It is then not hard to see how to generalize this process to create as large a collection of *moments* as we would like. Now you see why that long digression was necessary. The principal reason for taking the various moments is to remove the individual velocity components from the picture, to average over that portion of phase space, and therefore to obtain mean physical quantities that characterize the macroscopic spatial distribution of the matter. To do this within the limits of the function $f(\mathbf{x}, \mathbf{v})$, we proceed as follows.

If we integrate over the entire volume of phase space, we *must* recover the total number of particles in the system. That is,

$$N = \int_{\infty}^{\infty} d\mathbf{v} \int_{-\infty}^{\infty} d\mathbf{x} f(\mathbf{x}, \mathbf{v}) = \int d\mathbf{x} n(\mathbf{x}) \tag{1.1}$$

so if we are interested in keeping free the information about the spatial variations, that is, those depending on $\mathbf{x}$, we should restrict our integration to only the velocity components. We now have the prescription for taking moments! Assume that we have components $v_i(x, t) = V_i + u_i$ and that we are free to choose any such components for examination. The subscript is then a dummy, so that we can multiply these together as $f v_i v_j \ldots v_n$, which we can then integrate over the normalizable distribution function in velocity space.

The various moments are averages over the random velocity distribution function. Since we cannot measure this in detail, we get rid of it via integration (which is equivalent to averaging). Statistically, all of the macroscopic properties of the medium are the expectation values of the distribution and its moments. Historically, it was an important step forward when it was realized that the proper treatment of thermodynamics, namely the statistical approach rather than the vaguer mean-value methods of the mid-nineteenth century, could also be taken over into dynamics of media composed of individual randomly moving particles. It is no accident that the evolving theory of statistics closely paralleled – and spurred – the development of statistical mechanics. It provided a natural arena in which to display the ideas.

It is now time to begin taking the moments of the distribution. For example, the density is clearly (from the previous discussion and definition) the 0th moment. The momentum flux is the first, the energy the second. Note that *the moment need not be a scalar* – only $f(\mathbf{x}, \mathbf{v})$ was required to satisfy this condition. We can now clarify why an ensemble can be treated as a fluid and what it has to do with astrophysics. The choice is not forced on us by any *a priori* principles; it is just that we grow up with the voice of Newton ringing in our ears and it tells us how to evolve *these* particular quantities. We know that thermodynamics will enter at some stage in the evolution of the system and that therefore, since the kinetic basis of this subject is well established, we should try to build our equations to match those of the statistical approach. The thermokinetic approach gives us some insight into the evolution of the function f, and if the equations for the fluid are to have meaningful macroscopic form, we must be able to combine the two approaches. In order to do this, we must first write down some differential equation for the evolution of the distribution function and then see if the successive moments will give us what we are hoping for – a full-scale dynamical system for the observables.

Rather than present a full-blown derivation of the evolution equation based on the ergodic principle, let us proceed more heuristically. In the end it will get us to the same place, and this way is more of a *royal road* approach. For once, it is appropriate. Once we have arrived at the fluid equations, the distribution function will no longer play an important role. But it is important that we go through these steps, because stars in galaxies can be treated as if they were a fluid just as a gas can be, and the derivation of the stellar hydrodynamic equations involves much the same logic. The simplest, almost schematic, equation we can write down for the evolution of the distribution function is

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + \dot{v}_j \frac{\partial f}{\partial v_j} = \left(\frac{df}{dt} \right)_c \tag{1.2}$$

where we have included the possibility of reshuffling effects into the collisional term, $(df/dt)_c$. This term is really just a symbol, holding the place of a more complex, and more complete, expression that we will not need to write down. It describes the cumulative effect of collisional redistribution of particles among the different coarse-grained bins of the distribution function. As such, it is essential to the understanding of the microphysics but is averaged out when we take moments and is not important in the macroscopic problems we will face here. Put another way, this evolution equation is a way of describing the transfer of particles within the phase space. On the macroscopic scale – that is, where we the observers live – we see only the average quantities. For instance, if there is a bulk motion of the center of mass this is a mean velocity, where the *mean* is what results when we average out all the random motions. But even if the mean value of the velocity is zero, there is still a dispersion because of the random motions that we will call the internal kinetic energy density.

This equation is known as the *Boltzmann equation* and is the central one for the evolution of the distribution function. The collisional term as first described by Boltzmann is an integral term, which makes this highly nonlinear equation (when all effects are included properly) an integro-differential equation *for the distribution function, not for the dynamical variables as such*. The velocities and accelerations are for the peculiar (random) velocities, not the mean observables, and therefore must be included in any integration over the system. The right-hand side of the equation is a roulette machine of sorts. It shuffles particles around in the phase space by collisions of different *varieties* (Boltzmann's phrasing) and is, in effect, the cause of the entropy of the system. Note that this is the point of the famous H-theorem.[1] The definition of H and its evolution equation is through the right-hand side of Eq. (1.2).

1) I will add the historical point that this is not the Latin letter H, but rather the Greek capital eta. The theorem was meant to characterize entropy. Once you know this, it may place the development of the evolution equation in a broader context.

Although neglecting the collision term may seem rash, since in a fluid we are presented with an inherently chaotic collisionally dominated system, we can begin to make some limited progress. One way of looking at it is this. If the system has reached equilibrium, there will likely be as many collisions that transfer particles into as out of a particular volume of phase space. Thus the term on the right-hand side has no net effect. If we then restrict out attention to *equilibrium* states, we will not be far from the mark when we ignore this collisional term. A new name is now applied to the equation, the *Vlasov* equation. This is a small price to pay for gaining a tractable system of equations. Actually, it is even more than that: the equation is now *exact*. That is, our equation has the form

$$df(\mathbf{q}) = \frac{\partial f}{\partial q_i} dq_i$$

where we have assumed that the distribution function is dependent on the variables $\mathbf{q}$ and that the variables are summed over repeated indices. This is called the *Einstein* convention, and it will be employed throughout this book to economize the notation.[2] *If* the equation vanishes, then the variables $\mathbf{q}$ are not linearly independent, and so the equation can be called exact and has *characteristic solutions*. The Vlasov equation (as distinct from the full Boltzmann equation) satisfies this condition easily. The characteristics exist and are derived from

$$dt = \frac{dx_i}{v_i} = \frac{dv_i}{\dot{v}_i} \tag{1.3}$$

and it is this system which defines the constants of the motion – the constants on which the distribution function will depend. The function f is therefore given by

$$f(x_i, v_i; t) = f\left(x_i - \int v_i dt, \frac{1}{2} v_i v_i - \int \dot{v}_i dx, v_i - \int \dot{v}_i dt\right) \tag{1.4}$$

where the last term is trivial, and the first term is for translations along the particle trajectory and is also trivial. The middle term is the most interesting, since we see that this is the energy of the system and the one term which is connected with the moments of the equation. We now show that there is an easy, and historically interesting, way of arriving at the precise form of this function if the characteristics are known.

2) This convention is as follows. For a scalar product, $\mathbf{u} \cdot \mathbf{v} = \Sigma_i u_i v_i$. For a matrix product, $\Sigma_j A_{ij} x_j \rightarrow A_{ij} x_j$. The summation, unless explicitly stated, is over repeated indices. The trace is $\mathrm{Tr}(A) = A_{ii}$, while A_{jj}, no summation, indicates a diagonal component.

A Quick and Dirty Derivation of f

We have just said that f is a function of the characteristics $x_i - v_i t$, $v_i^2 - \dot{v}_i x_i$, and the trivial characteristic $\dot{v}_i t - v_i$. Is there anything else that we might use to specify the *form* of the function? In fact, we can do this by thinking for a moment about phase space. The distribution function for any cell in this space can be called f_i. For the space, the distribution function should be given by

$$f(\mathbf{x}, \mathbf{v}) = \prod_{i=1}^{N} f_i(x_i, v_i) \tag{1.5}$$

The function must depend on velocity, must be spatially homogeneous, and must be a scalar in the velocity in each of the spaces. In addition, it must be normalizable. One additional point should be noted:

$$f(\Gamma_1 + \Gamma_2) = f(\Gamma_1) f(\Gamma_2) \tag{1.6}$$

that is, if we have a phase space composed of two subunits, the total distribution function should be formed from the product of the individual functions. The simplest function which satisfies all these restrictions is *an exponential in the energy – that is, in the square of the velocity*! It is one of the characteristics and thus is a conserved quantity in phase space. Therefore, the distribution function has the form

$$f d\Gamma = \prod_i e^{-v_i^2/2\sigma_i^2} d\Gamma_i \tag{1.7}$$

where $d\Gamma_i \sim dv_i dx_i$. In other words, we get the Maxwellian velocity distribution. This derivation is the one first presented by Maxwell in his discussion of the rings of Saturn in his Adams Prize essay; he based it on some ideas by John Herschel concerning the normal distribution function in probability. In fact, it is the right choice for a thermal distribution and, since we neglect here effects of degeneracy, the right one for us as well. The Vlasov equation makes it clear why the distribution function has the particular dependences it does. Or, put differently, through the Vlasov equation we find the trajectories along which the information about the distribution function will propagate through the phase space.

1.2.3
Continuity and Momentum Equations

So far, we have learned only that there are constants of the motion (for each dimension), a fact we can read off Eq. (1.4). Now, we take the first moment by multiplying Eq. (1.2), without the collision term, by v_i to obtain

$$v_i \frac{Df}{Dt} + v_i \dot{v}_j \frac{\partial f}{\partial v_j} = 0 \tag{1.8}$$

using the convective derivative, defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} \tag{1.9}$$

We use the definitions of the moments and integrate by parts (remember: *in phase space, the velocities and accelerations are independent variables, and f does not depend on the accelerations*). Thus, we obtain

$$\rho \frac{DV_i}{Dt} = F_i \tag{1.10}$$

where now F_i is the force density. Amazingly enough, we now have regained the equation of motion. What would have happened if we had just gone ahead and taken the 0th moment? Try it! The result is

$$\frac{Dn}{Dt} + n\frac{\partial V_j}{\partial x_j} = 0 \tag{1.11}$$

In other words, we get the *continuity equation*! The point should then be clear that the evolution equations are those for the conservation conditions on the macroscopic observables. How did we get rid of the troublesome term

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial v_j}(f v_i v_j \ldots) d\mathbf{v}?$$

By recalling that the function f is defined as integrable and normalizable. Consequently, it vanishes at $+\infty$ and $-\infty$. Thus, the terms which are integrated *under the derivative operator* vanish symmetrically. The term obtained from the derivatives with respect to $\dot{v}_i$ is also easily dealt with. It can be integrated by parts:

$$v_i \frac{\partial f}{\partial v_j} = \frac{\partial}{\partial v_j}(f v_i) - f\delta_{ij} \tag{1.12}$$

Here we have used the fact that in the coordinate system we have chosen, v_i and v_j are orthonormal. This greatly simplifies the calculation since terms like $F_j \delta_{ij} \rightarrow F_i$ replace the higher-order terms.

A word on procedure is now in order. We assume that the distribution function depends only on the random components of the velocity field. Consequently, we can use *either* **u** or **v** as the integration variable. The mean velocity does not need to enter the calculation.

Let us step back for a moment and examine one of the terms in the derivation. We have taken

$$v_i v_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j}(f v_i v_j) - f \frac{\partial}{\partial x_j}(v_i v_j)$$

On integrating over the peculiar velocities, we see that certain terms in the expansion will fall out. To see this, notice the expansion of the first derivative term:

$$f v_i v_j = f(V_i V_j + V_i u_j + V_j u_i + u_i u_j)$$

so that, since the integral is taken over the random velocities, we have

$$\int v_i v_j \frac{\partial f}{\partial x_j} d\mathbf{v} = n \frac{\partial}{\partial x_j} (V_i V_j + \langle u_i u_j \rangle) \tag{1.13}$$

since the mean of the random component is $\langle u_i \rangle = 0$. There is, of course, no mean for the divergence, hence the second term in the expansion vanishes. This is just an expanded version of what we had seen before for the derivation of the equation of motion, but it makes clear a very important point connected with the Boltzmann equation.

To write the equations of motion and continuity in a more compact component form, we look at the individual components. Now we will revert, for a moment, to lower case representation for the velocity components, since we will be using this through the rest of the book. We have as the equation of motion for a force given by a potential field Φ:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \tag{1.14}$$

Again, we sum over repeated indices. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} \rho v_j = 0 \tag{1.15}$$

Combining the two, we obtain:

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial T_{ij}}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_i} \tag{1.16}$$

We have introduced the tensor T_{ij}, called the *stress tensor*. It is symmetric (since one of the variables is a dummy for summation and the other is a free choice) and is defined by

$$T_{ij} \equiv \rho v_i v_j + p \delta_{ij} \tag{1.17}$$

the sum of the mean kinetic energy (although not diagonal, this is the Reynolds stress we will use in later chapters) and an isotropic random term that we have called the pressure. This way of writing the equations is especially useful. The primary difference among the various treatments of hydrodynamic problems is geometry. The choice of a coordinate system can dramatically alter both the formalism and results for the equations of motion and the conservation conditions. By resorting to the most general representation, we can get around a lot of the specific differences and concentrate on the most general results. This is an especially convenient form for considering the forces on a body. For a steady flow (one for which the time derivative vanishes), the net force is given by the integral of the stress over the area of the body $F_i = \int T_{ij} dS_j$.

The stress tensor contains the Bernoulli term (see Section 1.8), which, for steady-state flow, must be divergenceless. It is then correct to say that it is the divergence of this term which drives the evolution of the flow in nonsteady-state conditions. In the discussion of accretion disks (Chapter 8) we will have recourse to this equation in its most useful form. For now, we merely note that the equation written in this form makes the generalization to different coordinate systems especially easy to understand, since we have a "machine" which will allow us to take the coordinate transformations into account in a natural way.

We began with the 0th moment. Here, we simply integrated the Boltzmann–Vlasov equation directly and obtained a term which depended on velocity. In the first moment, we have obtained second-order terms in the velocity. Clearly, by induction, with every higher moment the nth moment will always contain a term which is the mean of $n + 1$ components in the peculiar velocity. This is a real conundrum – and one which is at the core of all proposed solutions to the Boltzmann equation – the so-called closure problem. We have seen that the means are of the square of the peculiar velocity in the case of the first moment. This is the same as the velocity dispersion squared, so that the random motion is assumed to have a nonvanishing *mean square*. Now, if you think about a Gaussian for a moment, you will realize that it is a function which is characterized by three properties: it has a mean value of zero, a nonzero dispersion, and integrates to unity. No wonder the Boltzmann equation in its "Vlasovian" incarnation winds up having a Gaussian velocity distribution as its most direct solution. It is the simplest that preserves the stochastic properties of the distribution function.

Before we complete the system of conservation laws with the addition of the energy equation, we can already look at the virial theorem and its consequences. It is interesting to note that this theorem was originally defined for bodies for which only the motions, and not the energies, were known.

1.3
The Virial Theorem

Let us now stop taking the moments, short of having in hand the full three conservation conditions, and ask: what happens if we take the first two *spatial* moments? After all, phase space is a $6n$-dimensional hyperspace – three spatial and three velocity components. If we can take the velocity moments, why not also examine what happens in space?

A more precise way of stating our aim is this. Suppose now that we have been able to average over the velocity portion of the distribution function, we seek truly global properties of the body in question. Are there any *generalized* constraints that the *entire* distribution of matter must obey, regardless of the individual components and their particular situation in space? Several examples immediately come to mind. For instance, for an isolated body the total energy is constant, and the shape of the body and all of its other properties must be consistent with this fact. This is not a spatially dependent statement, it is holistic. As we shall shortly observe, a number of such statements apply to mechanical systems, fluid or otherwise, and these will be important in understanding some of the constraints that structure astrophysical flows. In effect, given the Vlasov equation we can take simultaneous moments of the form

$$\int_{\mathbf{x}} \int_{\mathbf{v}} x_i \cdots x_n v_i \cdots v_n (L_{\mathrm{Vlasov}} f(\mathbf{x}, \mathbf{v})) d\mathbf{v} d\mathbf{x} = 0 \tag{1.18}$$

where the integrals are taken over all possible spatial and velocity components and values, and L_{Vlasov} is the linear operator on f [Eq. (1.2)]. By so doing, we have a function that will depend only on time, the last variable over which we have not taken any averages. And even this situation can be altered if we perform time averaging. Since the first velocity moment gives us the equation of motion, the first spatial moment will yield important information about the energy.

To show this, we take the first spatial moment of Eq. (1.14) with respect to x_j (we will assume the reference position is $\mathbf{x} = 0$):

$$x_j \frac{DV_i}{Dt} - x_j G_i = 0 \tag{1.19}$$

where G_i is now a generalized force.

Integrating over all x, we can derive a tensor (or in the case of the ith moment a scalar) equation for the energy of the system. This is most easily seen by recalling that $v_i = \dot{x}_i$ and so integration by parts and using the definition of the mass in terms of the density will give an equation for the kinetic and potential energies and the moment of inertia. If we were to work strictly with the scalar form of the equation and neglect any of the explicitly time-dependent terms, we would wind up with a familiar equation:

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2T + W \tag{1.20}$$

which is the *virial theorem*. It is a constraint on the equilibrium state of the *whole* system, a statement of how the energy is redistributed to different parts of the system if any changes are made. Notice that the theorem provides a means for connecting bulk changes in the distribution of matter in the system with variations in the kinetic and potential energies. Here T and W are given by integrals over the mass of the system (found from the definition of the density used in the moment equations and noting that $\int \rho d\mathbf{x}$ is the mass of the system, $T = \int(\frac{1}{2}\dot{x}_i\dot{x}_i)dm$ is the kinetic energy, $W = \int x_i G_i dm$ is the potential energy, and $I = \int x_i x_i dm$ is the moment of inertia.

We should pause here for a moment and examine the consequences of this theorem for equilibrium configurations. This is, without a doubt, a most central result for astrophysics and therefore deserves further reflection. If we take a system which is in a state of gravitational equilibrium, as defined by stating that W is the gravitational potential, then the kinetic energy is easily found. If this configuration is a spherical particle distribution (see the discussion of the Lane–Emden equation in the chapter on similarity solutions), we can derive a velocity dispersion for the system using Eq. (1.20). In astrophysical environments, however – for instance the stars or galaxies in a cluster – we often measure only the velocity dispersion and not the total mass directly. By judicious application of the virial theorem, we can derive a mass. It should be noted that this is the mass required for the system to have such and such a velocity dispersion *in equilibrium, that is, after relaxation*. Recall that we stated that the moments are taken relative to the Vlasov equation and therefore assume that the distribution function satisfies that equation. Such distribution functions are in equilibrium and therefore completely related. Put differently: in order for the virial theorem result to yield a physically useful number, rather than merely a constraint, we must *assume* that the medium is already behaving as a fluid, as we have defined it in this chapter.

We now look in more detail at the meaning of the term we have called W. In the case of the momentum equation, we were able to show that there exists a term which is the pressure, as long as the medium has a velocity dispersion (in other words, as long as the medium is hot, since the temperature is the measure of the dispersion). Now we can make use of this explicitly. We separate out two terms in W, that which depends on external fields, which we shall now call simply Ω (to keep in line with the standard notation), and the pressure. The momentum equation has a term of the form:

$$\int x_i \frac{\partial p}{\partial x_i} dV = \int \frac{\partial x_i p}{\partial x_i} dV - 3\int p\, dV = \int p n_i dS_i - 3\int p\, dV \qquad (1.21)$$

where we have obtained the factor of 3 by taking the divergence of x_i (prove this for yourself) and the reduction of the first term to a surface integral by $\int \nabla \cdot \mathbf{Q} dV = \int \mathbf{Q} \cdot d\mathbf{S}$. The surface integral in Eq. (1.21) is assumed to vanish in the absence of surface tension or related terms. Do not forget from here on that *stars have no walls*. Now we specify that the body forces come from the gravitational potential, the energy of which is identified as Ω. Therefore, we have

$$2T + \Omega - 3\int p\,dV = 0 \tag{1.22}$$

for the virial theorem with pressure included explicitly. You have seen this last term before – it is simply the thermodynamic representation of the work done by the system under compression or expansion. In the absence of ordered motion, then the equation relates the action of external forces (or as we shall see later of gravity) to the work done by the system. If there is a thermal and ordered motion as well, then the virial equation is the full form just derived.

The total energy for a system undergoing no external compression (or in other words in equilibrium) is

$$E = T + \Omega \tag{1.23}$$

and therefore the binding energy, which is always negative, is given by

$$E = +\tfrac{1}{2}\Omega \tag{1.24}$$

In order for the system to be bound and stable in the presence of a potential, the total energy of the system must be negative. It can be shown (and will be later) that the second variation in the energy is $\delta^2 E < 0$ and that the first variation is $\delta E = 0$ in order for hydrostatic equilibrium to hold. These are the most basic equations of structure applied to fluid configurations which are gravitationally bound and also the source of the so called *negative specific heat* behavior of self-gravitating systems.

1.3.1 Higher-Order Virial Equations

One more digression before we return to the process of taking moments. We should briefly examine some generalizations of the virial theorem to higher order. If we instead take the moments in a more general way, allowing for the time dependence and also for the off-diagonal terms (terms of the form $x_i x_j$), we get a so-called higher-order virial theorem of the form

$$\frac{1}{2}\frac{d^2}{dt^2}I_{ij} = 2T_{ij} + W_{ij} \tag{1.25}$$

It should be obvious how we have generalized the result. The only reason for adding this is that the virial theorem applies to any dynamical system, regardless of the number of dimensions or the inherent symmetry of the system. Thus it should be possible to take the moments with respect to off-diagonal elements of nonspherical or nonsymmetric configurations and look for the *normal modes* by taking all time dependences to be periodic and varying as $e^{\omega t}$.

Then we get an eigenvalue equation:

$$\tfrac{1}{2}\omega^2 I_{ij} + 2T_{ij} + W_{ij} = 0 \tag{1.26}$$

We shall make some use of this much later, in the discussion of stability problems, but it is useful to show it here for a start, since it is an immediate and clever generalization of the usual mechanical virial. It should be added, by the way, that even more generalized virial theorems are possible if moments of the virial itself are taken (Chandrasekhar 1966, 1967; Lebowitz 1996). That is, in analogy with the velocity moments, we can take higher-order moments of the spatial components as well – nothing but timidity stops us.[3]

1.4 Energy Conservation

Once we have completed the job of getting the equations for the dynamics of the fluid, we must come to grips with the fact that any fluid which is hot and/or radiating or conducting heat will be subject to a tendency to cool. This is a simplistic way of saying it, perhaps, but the point is that any hot matter in an open world will tend to a state of lower temperature and somehow this should be expressed within the context of the equations with which we have been dealing. Of course, it should be clear that if it can be derived from the Boltzmann–Vlasov equation, we should be able to do it using moments. In fact, this is also more easily seen by considering the following point.

In the introductory part of this chapter, we wrote down the grammatical equivalents for the various moments, namely the density, momentum, and energy, and also defined the fluxes accordingly. Now the time has come to make use of the third one of these – the second moment. Take the moment of the Vlasov equation in such a way as to form a scalar. That is, take the moment

3) The equations of motion in tensor form provide the simplest way of seeing how to generalize the virial theorem. If the stress tensor is T_{ij}, then

$$x_i\left(\frac{\partial p v_i}{\partial t} + \frac{\partial}{\partial x_j}T_{ij}\right) = -x_i\rho\frac{\partial\Phi}{\partial x_i}$$

Therefore, we have

$$\frac{\partial^2}{\partial t^2}\int x_i x_i dM - 2\int v_i v_i dM + \int T_{ij}x_j n_i dS - \int \delta_{ij}T_{ij}d\mathbf{x} = \int \Phi\rho d\mathbf{x}$$

Here $dM = \rho d\mathbf{x}$ is the mass. This result can be generalized by taking the products with off-diagonal components (x_j). Now the surface term is also explicitly included so you can see the effects of boundary conditions on the virial theorem result.

of L_{Vlasov} with $v^2 = v_i v_i$. Therefore, we have

$$v_i v_i \frac{\partial f}{\partial t} + v_i v_i v_j \frac{\partial f}{\partial x_j} + v_i v_u \dot{v}_j \frac{\partial f}{\partial v_j} = 0 \tag{1.27}$$

The velocity is again composed of mean and random components. Remember that we are going to take the integrals only over the stochastic components. First, if we separate these components out, it will be easier to see what's going on. The product $v^2 = V^2 + 2u_i V_i + u^2$, where u is the thermal component. The advective term becomes

$$v^2 v_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j}(V^2 + 2u_i V_i + u^2)(V_j + u_j) f - f \frac{\partial}{\partial x_j}(v^2 v_j) \tag{1.28}$$

so that only the first term remains. But on taking the integral, we see that only the correlated terms remain: $\langle u_i u_j \rangle = \langle u^2 \rangle \delta_{ij}$. That is, for the divergence we have only the isotropic terms remaining. This is because we assume chaos and that the random components u are uncorrelated unless they are in the same direction. The last term becomes

$$v^2 \dot{v}_j \frac{\partial f}{\partial v_j} = \dot{v}_j \frac{\partial}{\partial v_j} f(V^2 + u_i V_i + u^2) - \dot{v}_i (V_i + u_i) f \tag{1.29}$$

since again $\partial v_k / \partial v_j = \delta_{kj}$. Only the mean velocity contributes to the last equation. Since the force is presumed to be velocity independent, the integral over the distribution function vanishes for the first term. Collecting the final integrated values, we find that

$$\frac{\partial}{\partial t}\left(\rho \frac{1}{2} v^2\right) + \frac{\partial}{\partial x_j}\left(\rho v_j \left[\frac{1}{2} v^2 + \frac{p}{\rho}\right]\right) + F_j v_j = 0 \tag{1.30}$$

Now notice that the divergence appears again, but this time involving the quantity $\mathbf{F}_{\text{heat}} = \rho \mathbf{v}(\frac{1}{2}v^2 + \mathcal{E})$. Here $\mathcal{E}$ represents the internal thermal energy. This quantity, the energy flux, has the characteristic dimensions of energy per unit area per unit of time and is the rate of transport of the energy out of the region in question. If the body is not isolated, there will be an additional term for the work, since g_j is the acceleration and therefore this will provide a work term for the bulk changes in the medium. We now perform the same tricks as before, taking the partial derivatives and integrating by parts, to obtain

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}(F_{\text{heat},j}) = 0 \tag{1.31}$$

where W is the work done by the system and consists of all of the thermodynamic functions, which we will discuss in the next section. In anticipation of the results for the thermodynamic variables, let us look at another form for the energy conservation equation. The kinetic energy density is $\frac{1}{2}\rho v^2$ and the internal energy density is ρE. The time derivative of the sum of these two energies, $d\rho\mathcal{E}/dt$, is

$$\rho\frac{d\mathcal{E}}{dt} + \mathcal{E}\frac{d\rho}{dt} = \rho\frac{d\mathcal{E}}{dt} - \rho\mathcal{E}\frac{\partial v_j}{\partial x_j} \tag{1.32}$$

where we have used the continuity equation, $\dot{\rho} = -\rho\nabla \cdot \mathbf{v}$, for the last step. Now:

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + \mathcal{E}\right) = v_j\frac{dv_j}{dt} + \frac{d\mathcal{E}}{dt} = -\frac{1}{\rho}v_j\frac{\partial p}{\partial v_j} + T\frac{dS}{dt} - \frac{p}{\rho}\frac{\partial v_j}{\partial x_j} \tag{1.33}$$

We have used (in anticipation of the next section) the relation that

$$\frac{dE}{dt} = T\frac{dS}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt} \tag{1.34}$$

where T is the temperature and S is the entropy. Collecting terms we therefore have

$$\frac{\partial}{\partial t}\rho\left(\frac{1}{2}v^2 + \mathcal{E}\right) + \frac{\partial}{\partial x_j}\rho v_j\left(\frac{1}{2}v^2 + \mathcal{E} + \frac{p}{\rho}\right) = \frac{d\epsilon}{dt} - \rho T\frac{dS}{dt} + \Lambda \tag{1.35}$$

where for compactness we have written $\rho\mathcal{E} = \epsilon$ as the total energy density. All of the loss terms are on the right-hand side, grouped into a term $\Lambda(T, \rho)$ that encapsulates all of the radiative, viscous, and dissipative processes that are not included in the Vlasov equation. If this term vanishes, we obtain the normal form of the heat equation. If the medium is at rest so that v vanishes, then the energy derivative is the change in the internal energy, $\dot{\mathcal{E}}$, which is a function of temperature, the heat flux is given by $-\kappa\nabla T$, the conductivity, and we obtain

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j}\kappa\frac{\partial T}{\partial x_j} \tag{1.36}$$

which is the normal form for the heat conduction equation, where κ is the heat conductivity (and we have not assumed that it is constant throughout the medium).

1.5 Some Thermodynamics

We now examine the thermodynamic variables in order to complete our picture of the fluid and its physical description. Let us start with the old but familiar first law of thermodynamics. It states that any work done on a thermal system produces a change in the internal energy and also in the heat. It also relates the entropy, S, and the state variables, namely the pressure, p, temperature, T, and volume, V. Put more formally, the first law states that the quasi-exact derivative of the heat function is given by

$$dQ = TdS = dE + p\,dV \tag{1.37}$$

Although you have certainly seen this many times before, it is useful here to note a few inter-relations among the variables which may come in handy later – especially because we see that this equation is essential to the definition of the equation of state. Let us write this last equation as

$$dQ = \frac{dE}{dT}dT + p\,dV$$

The change in the internal energy with respect to the temperature is the specific heat at constant volume – at least if the two extensive variables are T and V. We also know, from the equation of state for a perfect gas, that $pV = \mathcal{R}T/\mu$, where $\mathcal{R}$ is the gas constant and μ, is the mean molecular weight. Therefore, we get $dQ = (c_v + \mathcal{R}/\mu)dT - V\,dp$. From the fact that here we have taken the variation of the heat function with respect to T and p, we obtain

$$c_p = c_v + \mathcal{R}/\mu \tag{1.38}$$

and it therefore follows that $c_p/c_v > 1$. In fact, if we take a somewhat different form for the equation of state, in the case of $E = E(V,T)$ as a more general formalism, we get

$$c_p - c_v \left(\left(\frac{\partial E}{\partial V} \right)_T + p \right) \left(\frac{\partial V}{\partial T} \right)_p \tag{1.39}$$

We define a new variable, with which you will doubtless become quite familiar in the chapters ahead, that is, the ratio of specific heats $\gamma = c_p/c_v$. Notice that by Eq. (1.39), γ is always ≥ 1. In terms of the density, the first law is written as

$$dE = T\,dS + \frac{p}{\rho^2}d\rho \tag{1.40}$$

where the specific volume is $V = 1/\rho$. The enthalpy is given by $H = E + p/\rho$, which is also called the work function or the heat function given by

$$dH = T\,dS + \frac{1}{\rho}dp \tag{1.41}$$

Thus you see that the heat flux is really an enthalpy flux, and the energy conservation equation has a term $\boldsymbol{\nabla} \cdot \rho\mathbf{v}(\frac{1}{2}v^2 + H)$ for the flux divergence term.

1.5.1
More Virial Theorem Results

Let us now return to the virial theorem, for it has other manifestations that are really quite dramatic. Having defined γ not only helps in keeping the notation compact but also yields some insight into how the equation of state and the global equilibrium conditions interface. The internal kinetic energy is

$$\mathcal{J} = \tfrac{3}{2}(c_p - c_v)T = \tfrac{3}{2}(\gamma - 1)c_v T \tag{1.42}$$

where we have replaced $\mathcal{R}/\mu$ by $c_p - c_v$. The internal energy is $U = c_v T$, so that

$$3(\gamma - 1)U + \Omega = 0 \tag{1.43}$$

Now since the total energy is given by $U + \Omega$, we see that

$$E = -(3\gamma - 4)U = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega \tag{1.44}$$

The total energy of the configuration vanishes if $\gamma = \frac{4}{3}$. Keep this in mind, because shortly you will need it. In one very important respect the virial theorem leads to a most interesting astrophysical consequence. If a self-gravitating body contracts, the total energy increases. This is the *negative specific heat* problem that was mentioned earlier (see, e.g., Saslaw (1987) for an especially complete discussion of this phenomenon). You can think of this as the process that leads to gravitational collapse; that is, as a self-gravitating body cools it is forced to contract. This contraction heats it up, increases its energy losses (because of the increase in T the surface loss terms go up), and therefore the contraction rate increases to compensate for the losses. If the equation of state should be independent of temperature, as it is for a degenerate gas, the energy losses do not produce a contraction – the reason why white dwarf stars are static and main-sequence stars are not, for instance. Thus, our attention must be directed toward the equation that connects the thermodynamic variables, specifically the density and temperature, to the pressure, the equation of state.

1.5.2
Equation of State for a Polytrope

We now show that it is possible to write the equation of state in terms of the density variable and that we have a simple way of connecting the thermodynamics with the hydrodynamic equations. Take the case of an adiabatic medium by using

$$d\mathcal{Q} = c_v dT + \frac{\mathcal{R}T}{\mu V} dV \tag{1.45}$$

where the universal gas constant is $\mathcal{R}$ and μ is the mean molecular weight. If we set $d\mathcal{Q} = 0$ then

$$\frac{dT}{T} - (\gamma - 1)\frac{d\rho}{\rho} = 0\,, \quad \frac{dp}{p} - \frac{\gamma}{\gamma - 1}\frac{dT}{T} = 0\,, \quad \frac{dp}{p} - \gamma\frac{d\rho}{\rho} = 0 \qquad (1.46)$$

Note that these relations are written with the density rather than the volume (they are reciprocals). Notice that we are now able to obtain a closed-form solution for the pressure–density relation, the equation of state, provided all changes in the medium are adiabatic. This is not too stringent a restriction, however, because if the state of the gas does not change, then the equation of state will not either. Put another way, if the gas does not change its specific heats and behaves more or less ideally, then the equation of state will remain valid. Again, there are no radiative processes which make the system inherently open and thus preclude these definitions from applying throughout the medium. A simple form for the equation of state which will be useful in later work is

$$p = K\rho^{\gamma} \qquad (1.47)$$

where K is usually called the *entropy constant*. An equation of state of this form is called *polytropic*. It is one of the most generally used forms for the equation of state, being grounded firmly in the thermodynamic state of the medium. For a perfect gas, one composed of identical particles with only translational degrees of freedom and no additional correlations in the distribution function, the ratio of specific heats is $\frac{5}{3}$. This comes from the fact that the kinetic energy for each free motion is $\frac{1}{2}kT$; if internal degrees of freedom are available, such as rotation or vibration, γ is reduced. It should be noted that we have actually to deal with three possible adiabatic exponents. The basic polytropic equation is $p = K\rho^{\Gamma_1}$. The temperature equation is $\ln p = [\Gamma_2/(\Gamma_2 - 1)]\ln T$ and $\ln T = (\Gamma_3 - 1)\ln\rho$. For a perfect gas, the three exponents are equal, but in astrophysical fluids, which are especially prone to multiple ionization states and even the effects of degeneracy, these need not be identical. In fact, an equation for p that depends only on the density need not even result from a thermal process. The pressure in a degenerate gas comes from quantum statistics (the Fermi distribution) and is independent of temperature yet we can write $p \sim \rho^s$ for some power s (we don't need to concern ourselves with the value of s, it is enough to say temperature doesn't matter). Such pressure laws are called *barotropic* since they will produce pressure variations that precisely track those of density. We will use this again when discussing vorticity.

Since it is important to be able to derive the pressure laws for different types of fluids, and we have already examined the case of an ideal gas, let us look at one of the other kinds frequently encountered in astrophysical flows, a radiation-dominated gas. Remember that one distinguishing feature of astrophysical problems is the importance of radiation: in some cases, the energy density in the radiation can actually dominate over that of the matter, producing a much different type of gas than you will have encountered in the laboratory.

1.5.3 The Sound Speed

Here is the appropriate place to introduce a new dynamical variable, the sound speed. It is the speed with which a pressure disturbance propagates through a medium with an internal density ρ and pressure P. Because the pressure depends on the internal velocity dispersion of the gas, the sound speed is directly related to the temperature for a perfect gas. It is defined by

$$a_s = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} = \left(\frac{\gamma p}{\rho}\right)^{1/2} \tag{1.48}$$

for a polytropic equation of state. We will make much use of this quantity throughout this book, but will reserve the derivation of its connection with the speed of sound until the chapter on instabilities (Chapter 9).

1.5.4 The Equation of State for a Photon Gas

We have written out the equation for the entropy in a general way. Now, let us look at what happens for the adiabatic case in which we have an isotropic pressure resulting from a photon gas. We see that the entropy is an exact derivative, so that

$$\frac{1}{T}\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial T}\right)_V = \frac{\partial}{\partial T}\left(\frac{1}{T}\left[p + \left(\frac{\partial E}{\partial V}\right)_T\right]\right) \tag{1.49}$$

We now assert that the isotropic pressure is given by $\frac{1}{3}\epsilon(T)$, where $\epsilon(T)$ is the energy density, and that $E(T,V) = \epsilon(T)V$. Therefore, we have for the energy density

$$\frac{1}{T}\frac{d\epsilon}{dT} = -\frac{4\epsilon}{3T^2} + \frac{4}{3T}\frac{d\epsilon}{dT} \rightarrow \epsilon(T) = aT^4 \tag{1.50}$$

If we then take the adiabatic equation for a polytrope that was derived a moment ago, we see that a radiation-dominated gas is one which has $\gamma = \frac{4}{3}$. This is a very soft equation of state, in fact the softest that is permitted for stability of a self-gravitating fluid, and is due to the relativistic nature of the "particles" in the gas. Therefore, we have a simple way of connecting the equation of state with the thermodynamic state of the system. For astrophysical problems, this is most useful. We generally must deal with radiation-dominated fluids, or at any rate ones in which the radiation cannot be completely neglected.

1.5.5
Virial Theorem for Self-Gravitating Bodies

In light of the dynamical arguments we have just been through and the thermodynamic relations we have defined, let us collect a few relations that are of importance for cosmic bodies. The one common thread that makes these especially "astrophysical" is that they involve self-gravity of the object. For a body of constant mass, the internal structure is given by $dM(r)/dr = 4\pi r^2 \rho(r)$. The gravitational self-energy of a mass M of radius R is

$$E_{\text{grav}} = -G \int_0^M \frac{M(r)dM(r)}{r} \tag{1.51}$$

This becomes $-\frac{3}{5}GM^2/R$ for a uniform-density sphere. The total thermal energy content is

$$E_{\text{thermal}} = \int \frac{3}{2}kT(r)dM(r) \tag{1.52}$$

for a perfect gas. And the energy evolution equation becomes

$$\frac{d}{dt}\left(\frac{3}{2}kT\right) - \frac{p}{\rho^2}\frac{d\rho}{dt} = \dot{\epsilon} - \frac{1}{4\pi r^2 \rho}\frac{dL(r)}{dr} \tag{1.53}$$

where $\dot{\epsilon}$ is the rate of energy generation and $L(r)$ is the radiative loss, also called the luminosity. Equation (1.53) is also the basic equation for energy generation in a stellar interior.

1.6
Conservative Form of the Fluid Equations

For a perfect gas, we can collect the fluid equations in vector form. The continuity equation is written as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{v} = 0 \tag{1.54}$$

The momentum conservation equations become

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot (p\mathbf{I} + \rho \mathbf{v}\mathbf{v}) = -\rho \boldsymbol{\nabla} \Phi + \eta \nabla^2 \mathbf{v} \tag{1.55}$$

where $\mathbf{I}$ is the unit matrix. The (scalar) energy conservation equation becomes

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{v} \left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \Phi\right) = p\mathcal{L}(\rho, T) \tag{1.56}$$

where $\epsilon = \rho(\frac{1}{2}v^2 + E)$ is the total energy density, E is the internal energy, and $\mathcal{L}$ is a volumetric energy loss rate. The energy loss term takes into account all radiative and viscous loss terms (see Chapter 2 for the derivation of the Navier–Stokes term, which does not immediately follow from the Vlasov equation because we explicitly ignore the collisional term from the Boltzmann equation). These forms of the conservation equations are especially advantageous for our purposes because they show immediately the conditions that must be conserved within a flow. In addition, they have been shown to be convenient for numerical calculations.

1.7 Eulerian and Lagrangian Frames

This will be a short section. It assumes some experience on your part. Imagine that you are reading a map for someone who is driving and providing the directions for the route. Here individual style is important. Some, on giving directions, will say where to turn and where to go straight, giving distances between reference points and then locations of the critical places, as in "go 3 km north on Blvd. St. Michel and turn left at Blvd. St. Germain." Others would say, "continue for 3 minutes until Blvd. St. Germain comes up and turn left." This is essentially the difference between Eulerian and Lagrangian coordinates. One is in the frame of the external world, the other in the frame of the moving vehicle. Both are correct but, depending on the situation, both are not equally appropriate.

For a Lagrangian frame, the equations of motion take Newtonian form. That is, in this frame, because it is co-moving with the fluid, the time derivatives are ordinary and the advective term is absorbed. The coordinate **x** is a function of time, and velocities of external objects are referred to the frame of motion. In order to translate back into the stationary frame, a coordinate transformation has to be applied. Thus the Lagrangian forms for the fluid equations are

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \,, \quad \rho \frac{d\mathbf{v}}{dt} + \boldsymbol{\nabla} p + \rho \boldsymbol{\nabla} \Phi = 0 \tag{1.57}$$

In many astrophysical problems, the equations are solved instead in a frame depending on the mass. That is, you choose a convenient mass fraction (if the total mass remains constant) and, instead of solving for the mass at a specific radius, you solve for the radius at which that mass is located. The difference is that in time-dependent problems (as we will see in Chapter 9, this often provides a more useful frame of reference).

1.8 The Bernoulli Equation – The First Look

Now that we have defined both the method and the details of the process for a single equation of state, one in which we have assumed that there is only a perfect gas, it should also be clear that this can be generalized to include the cases in which radiation pressure, or even degeneracy, might be important. However, these are beyond the scope of this discussion and more properly dealt with in a course on the physics of stellar interiors. For completeness, however, a discussion of degeneracy is included in the appendix to this chapter. We shall later have recourse to the case in which the medium is radiation dominated. It should be noted, however, that even in this case it is possible to write down the equation of state in this form. It is useful to note that for a perfect gas, for which there are three degrees of freedom for the particles, the value of $\gamma = \frac{5}{3}$. For any gas in which there are more degrees of freedom (like a diatomic or polyatomic gas, in which internal states are available), the value of γ will decrease. Any ionization will also decrease its value. In Chapter 9 we shall discuss how this relates the thermal pool to the stability of a region.

Since we now have in hand an equation for the pressure as a function of the density, we can write down an additional description of a flow. Take the simple case of an adiabatic compressible fluid. The fluid is assumed to have no internal vorticity. This is because the advection term is

$$\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -(\boldsymbol{\nabla} \times \mathbf{v}) \times \mathbf{v} + \tfrac{1}{2} \boldsymbol{\nabla} v^2 \tag{1.58}$$

and we assume that $\boldsymbol{\nabla} \times \mathbf{v}$, the *vorticity*, vanishes (see Chapter 3). This also means that $\mathbf{v} = \boldsymbol{\nabla}\phi$, where ϕ is called the velocity potential (since the curl of this quantity vanishes identically everywhere). Such motion is said to define a streamline. Then by using Eq. (1.14) we get, for the equation of motion

$$v_j \frac{\partial v_i}{\partial x_j} + \gamma \rho^{\gamma-2} \frac{\partial \rho}{\partial x_i} = 0 \tag{1.59}$$

if there are no other body forces acting on the fluid. For a one-dimensional flow this becomes

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{constant} \tag{1.60}$$

which is the *definition* of Bernoulli flow. Named in honor of D. Bernoulli, this equation is one of the most basic tools for the analysis of flows. It states that if the internal velocity of the flow goes up, the internal pressure must drop. Therefore, the flow becomes more constricted if the velocity field within it increases. Another consequence of this equation is that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{constant} \tag{1.61}$$

which is the equation for a streamline ϕ. This particular parametrization of the velocity in terms of a potential field is especially useful for inviscid flows, because the streamlines define the motion of the fluid and permit the use of analytical techniques (like complex analysis and conformal representations) for solving the flow equations. We will not deal with these methods, however (but see Batchelor (1967), in the General Bibliography, for a very thorough discussion of their applications).

Instead, let us look at some of the implications of the Bernoulli equation. Think about the problem of liquid flowing down the side of a pitcher, and the fact that it will always seem to stick to the walls, and you will have a good feeling (albeit a wet one) for what is implied by this equation. In turn, it is the equation which is determined along a streamline to be a universal – it defines the characteristics of the flow and must be conserved along any streamline, by definition. Therefore, in the case of any fluid within a pressure-bounding medium, the flow is confined by the internal velocity and pressure against expansion into the background gas, and this equation also serves as the boundary condition for the flow. The enthalpy is defined by

$$H = c_p T + \frac{1}{2}v^2 = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}v^2 \tag{1.62}$$

which is essentially the same as the integration of the stress tensor for a one-dimensional flow. We can change the form of the equation for the one-dimensional system to

$$\frac{d\rho}{dv} = -\frac{\rho v}{a_\mathrm{s}^2} \tag{1.63}$$

Here we have represented the internal energy by $a_\mathrm{s}^2/(\gamma - 1)$; a_s is the sound speed. With this form of the equation, we obtain

$$\frac{dJ}{dv} = \rho\left(1 - \frac{v^2}{a_\mathrm{s}^2}\right) \tag{1.64}$$

for a simple flow for which $J = \rho v$. This equation gives the mass flux as a function of velocity. Clearly this function has a maximum at the sonic point and decreases for higher velocities. Let us assume that $J = J_\star$ at this point. We also assume that there is some point in the flow at which the velocity is zero. We can call this point x_0. Here we have for the flow an entropy S_0 and an enthalpy at x_0 of $H_0 = E_0$. The internal energy is given by $w \to c_p T = a_\mathrm{s}^2/(\gamma - 1)$. This results from the assumption of a perfect gas and the relation $c_p - c_v = \mathcal{R}/\mu$. Thus, for adiabatic flow, we obtain $S_\star = S_0$, and $E_\star + \frac{1}{2}v_\star^2 = E_0$, and that the velocity at the maximum is the local sound speed $a_{\mathrm{s},\star}$. The maximum velocity for the flow is therefore

$$v_\star = a_{\mathrm{s},0}\left(\frac{2}{\gamma - 1}\right)^{1/2} \tag{1.65}$$

and we find that the sound speed at the maximum is

$$\frac{a_{s,\star}^2}{\gamma - 1} + \frac{1}{2}a_{s,\star}^2 = \frac{a_{s,0}^2}{\gamma - 1} \rightarrow a_{s,\star} = a_{s,0}\left(\frac{2}{\gamma + 1}\right)^{1/2} \tag{1.66}$$

From this, we can obtain all of the thermodynamic variables as a function of velocity and therefore as a function of position in the flow. This is the result for an accelerating flow which transits the sonic point. It is a flow for which the cross-section is not included. We now look at what happens if we include the effect of an aperture of variable size through which the flow is directed.

1.8.1
The de Laval Nozzle: Bernoulli Flow with Confinement

Suppose we have, in addition to the Bernoulli equation, the continuity equation in a steady-state approximation (that is, the solution to the equation without time dependence):

$$C = \rho \mathbf{v} \cdot \mathbf{\Sigma} \tag{1.67}$$

where Σ is the area through which the flow is moving. We then take the area to be a function of position in the flow such that, for a one-dimensional flow, we have $\Sigma = f(x)$. Now we have, for a compressible fluid, the equation

$$\rho v \frac{dv}{dx} = -\frac{dp}{dx} \tag{1.68}$$

in the absence of external forces. We make the approximation for the equation of state that the velocity dispersion is a constant and that therefore we can write $P = \rho a_s^2$ where a_s is the sound speed. For now assume that it is a constant. Then, by the equation of continuity we have

$$v\frac{dv}{dx} = a_s^2\left(\frac{d\ln v}{dx} + \frac{d\ln\Sigma}{dx}\right) \tag{1.69}$$

We therefore see that there is a solution of the form

$$\tfrac{1}{2}v^2 - a_s^2 \ln v - a_s^2 \ln \Sigma = \text{constant} \tag{1.70}$$

so that, as the flow is constricted into a narrower aperture, it must move faster. This solution for the fluid flow has some remarkable consequences. In the case of a polytropic gas, where the density is a power law, we get

$$v\frac{dv}{dx} = -\frac{\gamma}{\gamma - 1}\frac{d\rho}{dx} \tag{1.71}$$

We assume that the ratio of specific heats remains constant throughout the flow. An alternative form for the equation of motion is

$$v\frac{dv}{dx} = -a_s^2\frac{d\ln\rho}{dx} - \frac{da_s^2}{dx}$$

so that we can remove the density by using the continuity equation:

$$(v^2 - a_s^2)\frac{1}{v}\frac{dv}{dx} = -\frac{da_s^2}{dx} + a_s^2\frac{d\ln\Sigma}{dx} \tag{1.72}$$

A remarkable result has just been obtained. We can relate the gradient of the sound speed to the velocity gradient via the gradient in the cross-section. In fact, we have the result that when the sound speed is reached in the flow, the equation for the sound speed gives

$$v = a_s \rightarrow 2\left(\frac{d\ln a_s}{dx}\right)_\star = \left(\frac{d\ln\Sigma}{dx}\right)_\star \tag{1.73}$$

We will have many recourses to the Bernoulli equation in the discussions that follow. This introduction is intended only to whet your appetite.

1.8.2 Jets: Introduction

Accretion disks are fine examples of astrophysics in two dimensions. They impose axial symmetry on their environment and introduce angular momentum into any material that happens to escape from their surface. It may not therefore be surprising that jets; axisymmetric and essentially one-dimensional flows, are one of the most ubiquitous phenomena in astrophysics. They are one-dimensional because the momentum is carried primarily *along* the axis of the flow. And they are axisymmetric because the azimuthal coordinate does not enter into the structure – well, it is not entirely unimportant. Because a jet is free to move in three dimensions, even if it does not in the unperturbed state, there are free modes along the surface and within the body of the jet that make the azimuthal coordinate very important. The engineering literature has contained work on jets since the first decades of the century. There was considerable interest in this means of propulsion, first in naval architecture and later in aerodynamics, as an alternative to propellers. Several reasons seem to have dominated. One was the problem of stability. Another was the efficiency of the momentum transfer possible with jet propulsion.

The astrophysical importance of jetlike flows was also recognized quite early. Curtis (1918)[4] described the *visual* observations of M87, the central elliptical Galaxy in the Virgo cluster, as a jet. This was elaborated by Minkowski and Baade in the 1940s. The Galaxy associated with the quasar 3C 273 also has an optical jet that was noted in the first observations in the early 1960s. Plume and jetlike phenomena were invoked to explain double lobed radio sources,

4) The references to jets are so numerous, and occur in so many places throughout this book, that they are included as a separate section in the general bibliography and will not be repeated at the ends of the chapters.

the first important attempt being by Blandford and Rees (1974). Much elaboration of the model, though, was spurred by a remarkable discovery: SS 433. This source was first observed by the Ariel V and SAS-3 x-ray satellites and quickly identified as a radio source. Its optical identification soon provoked astonishment (I do not use that term lightly). Optical spectra showed that the lines of the Balmer series and of neutral helium display time-variable velocities with an amplitude of $0.26c$. The lines are narrow, never show P Cygni structure and, while variable otherwise, act as a reliable clock (Margon 1984). Interferometric radio observations are also completely consistent with a precessing jet that occasionally belches out blobs (Hjellming and Johnston 1981). The discovery of bipolar outflows from some planetary nebulae and observations of jetlike structures connected with sites of active star formation, have made jets and their related phenomenology among the most familiar of all hydrodynamic phenomena in astronomy. There is too much written about the phenomena of radio jets to go into here, and this is generally specialized to radio observations of near- and ultrarelativistic flows for non-neutral plasmas. There is enough to do just to understand ordinary sluggish Newtonian flows. That is the purpose of this section.

For the purposes of our discussion, a jet will be defined as a confined fluid structure, bounded by a stable fluid and along which there is an essentially one-dimensional flow. This is an important definition to keep in mind because observationally, by definition, a collimated structure is called a jet. A lot often gets assumed. First, that there is a flow. This can be observed directly in one class of sources, namely bipolar flows and emission line jets seen in active star-forming regions of molecular clouds, and less directly in another, specifically SS 433. But in the more spectacular cases, radio galaxies, the direct evidence is lacking and more indirect arguments and analogies play the dominant role. There will not be much discussion of large-scale radio jets here, however. Observations point to the bulk motion in these jets as being relativistic. Because such problems would require more expansion than we have space available for, all but the most schematic properties of these will be ignored.

1.8.3 Basic Physics

An expanding jet is a fully three-dimensional object, at least in this world.[5] Jets can be treated as either, but the consequences of a two-dimensional treatment are quite severe when comparing the calculated structure to real astrophys-

5) George Abbott, whose book *Flatland* has been the source of much inspiration to generations of geometers and physicists, would certainly have understood the difference between saying that a jet is imbedded in a three-dimensional world and saying that it is a three-dimensional object.

ical objects. A two-dimensional jet is generated by many different physical conditions. For instance, a fluid issuing from a rectangular slit will enter the surrounding medium in a plane. It is an axisymmetric flow in the sense that it has a central axis with the maximum velocity and a finite momentum. A fluid leaving a wall-confined region also bears many of the same characteristics.

1.8.4
Subsonic Jets

Laminar jets present a classical laboratory example of the basic physics of axisymmetric unidirectional flows. For this reason, let us look at how the analysis of such objects proceeds and then generalize to more astrophysically interesting problems.

A two-dimensional jet is generated when a flow emerges from a slit or when it leaves the edge of a wall or plate. It also resembles a wake in many of its characteristics. Although such a jet is not a very good approximation of an astrophysical entity, it provides an example of closed-form solutions and serves to illustrate the basic physical problem. Also, in a sense, the axial jet is also two-dimensional, but the differences are very important between planar and cylindrical (or solid-angle) jets. Before generalizing to a truly axial jet, let us take a look at the planar one. Call the velocity component parallel to the x direction along the jet u. The momentum per unit volume is ρu carried by a symmetric jet (one symmetric on reflection about the midplane) and the rate of transport of momentum across a surface is defined by $\mathcal{M} = \int \rho v^2 dS$. For a two-dimensional jet, dS is replaced by dy, the line transverse to the jet. For an incompressible fluid $\nabla \cdot \mathbf{v} = 0$. The transport of momentum is constant along the direction of propagation. To see this take

$$\frac{d\mathcal{M}}{dx} = 2\rho \int_{-\infty}^{\infty} u \frac{\partial u}{\partial x} dy \tag{1.74}$$

and substitute the continuity equation. Since $\partial u/\partial y$ and v both change sign across the jet axis, the integral vanishes. Therefore $d\mathcal{M}/dx = 0$. The next assumption we can make is that the jet has a simple velocity law. The spreading away from the axis is assumed to be self-similar and to depend only on γ/δ, where δ is the width; thus $u(x,y) = u_0(x)f(y/\delta)$. Next, and this is merely for simplicity, we take the axial speed u_0 to vary as x^a and δ to vary as x^b. From the equation of motion, this implies that $U^2x^{-1} \sim \nu U\delta^{-2}$ so that $a - 1 + 2b = 0$. Then from the definition of $\mathcal{M}$, it follows that $2a + b = 0$. This gives a scaling law of the form $a = -\frac{1}{3}$ and $b = \frac{2}{3}$ so the opening angle of the jet is constant with a half-angle of approximately 33°.

The jet expands by plowing up material ahead of it, some of which is "absorbed" and some of which is deflected. Its surface exerts a force on the medium $\rho v_j^2 A$, where A is the surface area of the head. The medium in turn is acting to slow it down, so the velocity of the head into the external medium simply scales as $v_\mathrm{h} = (\rho_\mathrm{j}/\rho_0)^{1/2} v_\mathrm{j}$. It is very important to note, though, that this estimates only the rate of momentum transfer and neglects some very important dynamical effects.

We see that a jet carries momentum of a magnitude $\dot{M}u_1$ and a bulk kinetic energy $\frac{1}{2}\dot{M}(u_1^2 + \sigma^2)$, where σ is the random component estimated by the spectral linewidths or by random motions observed directly within the jet. It has a force of $\rho_\mathrm{j} u_1^2 A_\mathrm{j}$ on the background gas and stagnates when the background pressure, $\rho_0 a_{\mathrm{s},0}^2$ is of the same order. Therefore, to ensure that the jet does not stall, the Mach number for the jet must be approximately $M_\mathrm{j} \geq (\rho_0/\rho_\mathrm{j})^{1/2}$. Notice that for an overdense jet any supersonic flow, and even subsonic flows, will not be stopped simply by ram pressure. But for an underdense jet (a so-called *light jet*) the flow must be supersonic in order to ensure continued propagation.

Appendix: Langmuir Waves as Perturbations of the Distribution Function

We will not deal very much with particle kinetics or plasmas in this book. But it is impossible to pass up the opportunity to illustrate an example of how the distribution function can be used, with moment methods and the Vlasov equation so fresh in your mind. The other reason for this appendix is that it is the best way to introduce an important quantity, the plasma frequency, and also to illustrate how perturbations on the distribution function result in dispersion relations. All of these ideas will be important later on. The best arena for studying the uses of the Vlasov equation is in the context of dilute gases, in which the individual particle kinetics are important. Recall that the point of the Vlasov equation is that it permits the study of what collections of weakly interacting particles will do in the presence of background fields.

Consider the Vlasov equation including the effects of charged particles. Here the force comes from the generated E field. We must supplement the equation for the distribution function by using a field equation, in this case the Poisson equation:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{1.75}$$

where ρ is the charge density. This is the *net* charge, that is, the excess of the charge from a zero value. We can consider that the field gives rise to a charge separation and that, if the plasma initially has some velocity distribution function, which is the same for the ions and electrons, the application of a background field will produce a change in the two velocity distributions so that the electrons will be "heated" relative to the ions.

In this way, the electrons understand that they are living in the same neighborhood, and their response to the imposed field is collective. It is through this response, governed by the Poisson equation, that the distribution function evolves. Otherwise the particles would respond individually and we would simply have a collection of collisionless orbits. But here, because once the distribution of velocities is changed in the gas it also feeds back into the long-range forces and the accelerations, there is a mechanism for changing the distribution function of the entire plasma. However, the electrons are not free to move, and they must drag the ions along as they are accelerated. This is because the plasma, unless there are constraints which prevent the free motion of the ions, will try to cancel the applied field and also to heal the rift between the electrons and the ions.

Since we have $\rho = e(n_\mathrm{i} - n_\mathrm{e})$, and the two densities are moments of the respective distribution functions, we have

$$\frac{\partial}{\partial t} f_s + u \frac{\partial}{\partial x} f_s + \frac{q_s}{m_s} E \frac{\partial}{\partial u} f_s = 0 \tag{1.76}$$

assuming that we have two species (s being i for the ions and e for the electrons). Further, $q_\mathrm{i} = e$ and $q_\mathrm{e} = -e$. We assume that we have a one-dimensional electric field placed across the medium, but that the distribution function is fully three-dimensional. Now assume that n_0 is the density of the undisturbed medium, so that

$$f_s(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_{1s}(\mathbf{x}, \mathbf{v}, t) \tag{1.77}$$

and that f_0 is the unperturbed distribution function. Further, assume that the electric field is a small perturbation. Perhaps it is a wave incident on the medium, or somebody turning on the potential across the plasma between two plates. Assume finally that all spatial and temporal terms are given by the Fourier amplitudes and that they vary with space as k and with time as ω where these are the wave number and frequency, respectively.

$$ikE(\omega) = 4\pi e n_0 \int_{-\infty}^{\infty} du\, dv\, dw (\phi_{1\mathrm{i}} - \phi_{1\mathrm{e}}) \tag{1.78}$$

Here we have assumed that

$$\phi_{1s} = \int f_{1s}(\mathbf{x}, \mathbf{v}, t) e^{i\omega t - kx} dt\, dx \tag{1.79}$$

The transform of the Vlasov equation becomes

$$(-i\omega + iku)\phi_{1s} + \frac{q_s}{m_s} E(\omega) \frac{df_{0s}}{du} = 0 \tag{1.80}$$

so that we can solve for ϕ_{1s} to obtain

$$\phi_{1s} = i \frac{q_s/m_s}{\omega/k - u} \frac{df_{0s}}{du} \tag{1.81}$$

Substituting this into the Poisson equation, we derive a simple relation for the *dispersion relation* for a wave in this plasma, strictly in terms of the kinetic distribution function:

$$-\frac{4\pi n_0 e^2}{k^2} \int d\mathbf{v} \frac{dg_0}{du} \left(\frac{1}{\omega/k - u} \right) = 1 \tag{1.82}$$

where we have abbreviated $f_{0\mathrm{i}}/m_\mathrm{i} - f_{0\mathrm{e}}/m_\mathrm{e}$, with $g(u)$. Since nature is constructed in such a way (unless we deal with a positron–electron plasma) that $m_\mathrm{e} \ll m_\mathrm{i}$, we can assume that the electrons dominate and therefore neglect the term involving $f_{0\mathrm{i}}$.

We now define the plasma frequency, a characteristic frequency for the oscillation of the separated charges, as

$$\omega_\mathrm{p}^2 = \frac{4\pi e^2 n_0}{m_\mathrm{e}} \tag{1.83}$$

so that we can rewrite Eq. (1.82) as

$$1 + \frac{\omega_\mathrm{p}^2}{k^2} \int d\,\mathbf{v} \frac{d f_{0\mathrm{e}}(u)/du}{\omega/k - u} = 0 \tag{1.84}$$

In order to avoid the pole at $\omega/k = u$, we assume that the phase velocity of the waves is large compared with the velocity of the electrons. First we integrate by pairs to remove the derivative of the distribution function from under the integral in Eq. (1.83). Now expand the denominator as a power series:

$$\frac{1}{(\omega/k - u)^2} \approx \left(\frac{k}{\omega} \right)^2 \left(1 + 2\frac{ku}{\omega} + 3\left(\frac{ku}{\omega} \right)^2 + \cdots \right) \tag{1.85}$$

The integral of the even powers does not vanish, because of the symmetry of $f_{0\mathrm{e}}$, so that, assuming that $f_{0\mathrm{e}}$ is normalized and that the second moment of the distribution function is σ_e, we get

$$1 - \frac{\omega_\mathrm{p}^2}{\omega^2} - 3\frac{k^2\sigma_\mathrm{e}^2}{\omega^4} = 0 \tag{1.86}$$

which is a quadratic in ω^2. The final dispersion relation, which relates k to ω, is

$$\omega^2 = \omega_\mathrm{p}^2 + 3\sigma_\mathrm{e}^3 k^2 \tag{1.87}$$

which is also known as the *Langmuir dispersion relation*. Note that it implies that the group velocity of the waves is different from the phase speed, such that the two are reciprocals of each other.

The reason for dwelling on this in the chapter on the Vlasov equation is that the methods used in the derivative of this equation are really quite general. With the exception of the specific equation for the electrostatic field E, we could have used the Poisson equation for a gravitational field. In this case, we would have replaced E by $\nabla\Phi$, the gradient of the gravitational potential, and used only the mass of the *single species* of particle rather than the charge and the ion–electron pairs. The same method is used for a Galaxy consisting of collisionless stars as for the plasma with which we have been dealing. Both are possibly unstable, and both have a characteristic dispersion relation for propagating disturbances. In the case of the self-gravitating medium, we shall return to the stability problem in Chapter 9. For the moment, keep it in mind as something coming down the road.

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