

## Introduction

### 1.1 Non-Newtonian Fluids

Fluids that show a departure from Newtonian behavior are termed as *non-Newtonian fluids*. When the flow behavior of a material follows the basic law of viscosity proposed by Isaac Newton, then it is said to depict Newtonian behavior. The constitutive equation for Newtonian fluids, which show constant viscosity without any yield stress and/or elastic component, is given as

$$\tau = \mu \dot{\gamma} \quad (1.1)$$

where  $\tau$  is the shear stress,  $\dot{\gamma}$  is the shear rate, and the constant  $\mu$  is termed as the Newtonian viscosity. In general, for determining velocity distributions and stresses, incompressible Newtonian fluids at constant temperature can be characterized by two material constants, namely, the shear viscosity  $\mu$  and the density  $\rho$ . Once these quantities are measured, the velocity distribution and the stresses in the fluid, in principle, can be found for any flow situation. In other words, different isothermal experiments on a Newtonian fluid would yield a single constant material property, namely, its viscosity  $\mu$  whose units are milliPascals seconds (mPa s). Some examples of the commonly known Newtonian fluids are water ( $\mu \approx 1$  mPa s), coffee cream ( $\mu \approx 10$  mPa s), olive oil ( $\mu \approx 10^2$  mPa s), and honey ( $\mu \approx 10^4$  mPa s).

Any material that does not behave rheologically in accordance with the described behavior is termed as a non-Newtonian fluid. Detailed discussions relating to non-Newtonian fluids are available in a number of books (Wilkinson, 1960; Skelland, 1967; Astarita and Marrucci, 1974; Darby, 1976; Schowalter, 1977; Dealy and Wissbrun, 1990; Macosko, 1994; Shenoy and Saini, 1996; Larson, 1998; Shenoy, 1999; Morrison, 2001; Chhabra and Richardson, 2008; Goodwin and Hughes, 2008; Irgens, 2013; Mezger, 2014; Osswald and Rudolph, 2015) as well as other review articles (Becker, 1980; Rosen, 1979; Rudraiah and Kaloni, 1990).

Non-Newtonian fluids can depict elastic, viscous, or viscoelastic behavior and exhibit one of the following features:

- a) Time-dependent viscosities at fixed shear rates;
- b) Shear-rate- dependent viscosities in certain shear rate ranges with or without the presence of an accompanying elastic solid-like behavior;
- c) Yield stress with or without the presence of shear-rate- dependent viscosities.

Purely elastic materials deform elastically upon the application of stresses such that the deformation is reversible and the energy of deformation is fully recoverable when the stresses are released. In purely viscous materials, any mechanical energy input into the system by the application of stresses is dissipated in the form of heat and cannot be recovered by releasing the stresses. Viscoelastic materials show response to deformation that lies in varying extent between those of purely elastic and purely viscous materials.

In reality, the same material can behave as purely elastic or purely viscous or viscoelastic during the deformation process. It is the time scale that decides how the material would behave rheologically. The relationship between the time scale of deformation to which a material is subjected and the time required for the material to respond determines whether the material behavior is elastic, viscous, or viscoelastic. The ratio of characteristic time for the material to respond to the time scale of deformation is defined as the Deborah number by Reiner (1949, 1960, 1964)

$$De = \frac{\lambda_c}{\lambda_s} \quad (1.2)$$

where  $\lambda_c$  is the characteristic time and  $\lambda_s$  is the time scale of deformation.

The characteristic time,  $\lambda_c$ , for any material can be defined as the time required for the material to reach 63.2% or  $[1 - (1/e)]$  of its ultimate retarded elastic response to a step change.

If  $De > 1.0$ , elastic effects are dominant; whereas if  $De < 0.5$ , viscous effects prevail. For any values of Deborah numbers other than these two given extremes, the material would depict viscoelastic behavior.

### 1.1.1 Non-Newtonian Viscous Behavior

Materials that show non-Newtonian behavior without an elastic response are said to depict non-Newtonian viscous behavior. Thixotropic, rheopectic, pseudoplastic, dilatant, Bingham plastic, and pseudoplastic with yield stress behaviors are manifestations of non-Newtonian viscous behavior.

#### 1.1.1.1 Thixotropic Behavior

In the case of materials showing thixotropic behavior, the shear rate is a function of the magnitude and duration of shear as well as a function, possibly of the time lapse between consecutive applications of shear stress. These materials exhibit a reversible decrease in shear stress with time at a constant rate of shear and fixed temperature. The shear stress, of course, approaches some limiting value.

#### 1.1.1.2 Rheopectic Behavior

Materials showing rheopectic behavior exhibit a reversible increase in shear stress with time at a constant rate of shear and fixed temperature. At any given shear rate, the shear stress increases to approach an asymptotic maximum value.

#### 1.1.1.3 Pseudoplastic Behavior

In the case of materials showing pseudoplastic behavior, the shear rate at any given point is solely dependent upon the instantaneous shear stress, and the duration of shear does not play any role so far as the viscosity is concerned. These materials depict a decrease in viscosity with increasing shear rate and hence are often referred to as shear-thinning materials.

#### 1.1.1.4 Dilatant Behavior

Materials that show dilatant behavior depict an increase in viscosity with increasing shear rate and hence are often referred to as shear-thickening materials.

#### 1.1.1.5 Bingham Plastic Behavior

Materials exhibiting Bingham plastic behavior do not flow unless the stress applied exceeds a certain minimum value, referred to as the yield stress, and then show a linear shear stress versus shear rate relationship.

#### 1.1.1.6 Pseudoplastic Behavior with Yield Stress

Materials that exhibit pseudoplastic behavior with yield stress have a nonlinear shear stress versus shear rate relationship in addition to the presence of a yield stress.

### 1.1.2 Non-Newtonian Viscoelastic Behavior

Materials that show non-Newtonian behavior with the added feature of elasticity are said to depict non-Newtonian viscoelastic behavior. Such materials exhibit rheological properties which lie in between those of elastic solids and viscous liquids.

#### 1.1.2.1 Highly Elastic Behavior

Viscoelastic materials have a certain amount of energy stored in them as strain energy, thereby showing a partial elastic recovery upon the removal of a deforming stress. At every instant during the deformation process, viscoelastic materials try to recover completely from the deformed state but are unable to do so and lag behind. The lag is a measure of the elasticity or so-called memory of the material. In other words, by virtue of their viscoelastic nature, such materials have the ability to recoil. However, they do not return completely to their original state when deformed because of their fading memory. Viscoelasticity allows such materials to remember where they came from, but the memory of their recent configurations are always much better than those of their bygone past, thus lending them the characteristics of a fading memory.

The various types of non-Newtonian fluids along with typical examples are summarized in Table 1.1.

#### 1.1.2.2 Mildly Elastic Drag-Reducing Behavior

Fluids that show a near Newtonian viscous behavior accompanied by a mildly elastic behavior fall in a different class known as drag-reducing fluids. In a

**Table 1.1** Various types of non-Newtonian fluids.

Fluid type	Definition	Typical examples
<ul style="list-style-type: none"> <li>• Pseudoplastic</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that depict a decrease in viscosity with increasing shear rate and hence often referred to as shear-thinning fluids</li> </ul>	<ul style="list-style-type: none"> <li>• Blood</li> <li>• Filled polymer systems</li> <li>• Pharmaceutical preparations</li> <li>• Polymer melts</li> <li>• Polymer solutions</li> <li>• Printing inks</li> </ul>
<ul style="list-style-type: none"> <li>• Dilatant</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that depict an increase in viscosity with increasing shear rate and hence often referred to as shear-thickening fluids</li> </ul>	<ul style="list-style-type: none"> <li>• Aqueous suspension of titanium dioxide</li> <li>• Gum solutions</li> <li>• Starch suspensions</li> <li>• Wet sand</li> </ul>
<ul style="list-style-type: none"> <li>• Bingham plastics</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that do not flow unless the stress applied exceeds a certain minimum value referred to as the yield stress and then show a linear shear stress versus shear rate relationship</li> </ul>	<ul style="list-style-type: none"> <li>• Certain asphalts and bitumen</li> <li>• Jellies</li> <li>• Sewage sludges</li> <li>• Thickened hydrocarbon greases</li> <li>• Tomato ketchup</li> <li>• Toothpaste</li> <li>• Water suspensions of clay/fly ash/metallic oxides</li> </ul>
<ul style="list-style-type: none"> <li>• Pseudoplastic with a yield stress</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that have a nonlinear shear stress versus shear rate relationship in addition to the presence of a yield stress</li> </ul>	<ul style="list-style-type: none"> <li>• Heavy crude oils with high wax content</li> <li>• Filled polymer systems</li> </ul>
<ul style="list-style-type: none"> <li>• Thixotropic</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that exhibit a reversible decrease in shear stress with time at a constant rate of shear and fixed temperature. The shear stress, of course, approaches some limiting value</li> </ul>	<ul style="list-style-type: none"> <li>• Coal-water slurries</li> <li>• Crude oils</li> <li>• Drilling muds</li> <li>• Filled polymer systems</li> <li>• Mayonnaise</li> <li>• Salad dressing</li> <li>• Water suspensions of bentonite clays</li> <li>• Yoghurt</li> </ul>
<ul style="list-style-type: none"> <li>• Rheopectic</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids exhibit a reversible increase in shear stress with time at a constant rate of shear and fixed temperature. At any given shear rate, the shear stress increases to approach an asymptotic maximum value</li> </ul>	<ul style="list-style-type: none"> <li>• Some clay suspensions</li> </ul>
<ul style="list-style-type: none"> <li>• Viscoelastic</li> </ul>	<ul style="list-style-type: none"> <li>• Fluids that possess the added feature of elasticity apart from viscosity. These fluids exhibit process properties which lie in-between those of viscous liquids and elastic solids</li> </ul>	<ul style="list-style-type: none"> <li>• Filled polymer systems</li> <li>• Polymer melts</li> <li>• Polymer solutions</li> </ul>

Source: Shenoy (1999). Reproduced with permission of Springer.

number of practical fluid flow situations, energy losses near solid surfaces due to turbulent friction are of very high magnitude. Mildly elastic drag-reducing fluids are known to help in reducing the tremendous energy losses due to turbulent skin friction. Various types of additives in the flowing fluid have been found to be effective: (i) macromolecules like those of polymers, biological additives or surfactants, and (ii) solid-particle suspensions containing solids like fine grains or fibers. Most of the researchers in the field of drag reduction have worked with fluids at room temperature because of the practical importance of using drag reducers in ship-building industries, for fire-fighting operations, oil-well-fracturing processes, and so on, where high temperatures are not involved. The idea of using drag-reducing additives in central heating systems was explored by Shenoy (1976) in order to study the effectiveness of drag reducers at high temperatures. An epitome of the results of the tremendous work done on drag reduction can be obtained in a number of reviews and reports (Patterson *et al.*, 1969; Lumley, 1969, 1973; Gadd, 1971; Darby, 1972; Hoyt, 1972; Landahl, 1973; Fisher and Ash, 1974; Palyvos, 1974; Little *et al.*, 1975; Virk, 1975; White and Hemmings, 1976; Berman, 1978; Giesekus *et al.*, 1981; Sellin *et al.*, 1982b; Shenoy, 1984b; Berman, 1986; Wilson, 1988; Singh, 1990).

## 1.2 Rheological Models

The constitutive equations, which relate shear stress with shear rate, involve the use of two to five parameters. Many of these constitutive equations are quite cumbersome to use in heat transfer analyses and hence only those models that will be used later in the text are presented here. For a broader understanding of various other available models, one could refer to the section on rheological models covered in the books of Shenoy and Saini (1996) and Shenoy (1999).

### 1.2.1 Non-Newtonian Viscous Behavior in Laminar Flow

#### 1.2.1.1 Ostwald-de Waele Power-Law Fluid

For inelastic non-Newtonian fluids in laminar flow, the simple two-parameter power-law model originally proposed by Ostwald (1925, 1926) and De Waele (1923) and fully described by Reiner (1949) is the most popular and commonly used. The equation for this model is given as

$$\tau = K\dot{\gamma}^n \quad (1.3)$$

where  $K$  denotes the consistency index of the material, with higher values representative of more viscous materials, and  $n$  is the power-law index giving a measure of the pseudoplasticity, with departure from unity showing more pronounced shear-thinning or shear-thickening characteristics. The power-law index  $n$  basically represents the slope of the  $\tau$  versus  $\dot{\gamma}$  curve in the medium-to-high shear rate range.

### 1.2.1.2 Sutterby Fluid

In this case, shear stress–shear rate data are described by

$$\tau_{xy} = -\mu_{app}\dot{\gamma} \quad (1.4)$$

with

$$\mu_{app} = \mu_0 \left( \frac{\operatorname{arcsinh} B' \dot{\gamma}}{B' \dot{\gamma}} \right)^{A'} \quad (1.5)$$

In the limit of low shear rates  $\dot{\gamma} \rightarrow 0$ , this model predicts that  $\mu_{app} \rightarrow \mu_0$ . At higher shear rates, a shear-thinning viscosity  $\mu_{app}$  decreasing with increasing  $\dot{\gamma}$  is predicted.

### 1.2.1.3 Ellis Fluid

For this type of non-Newtonian inelastic fluid, shear stress–shear rate data are correlated using the following expression for the apparent viscosity

$$\mu_{app}^{-1} = \mu_0^{-1} [1 + (\tau_{xy}/\tau_{1/2})^{y'-1}] \quad (1.6)$$

As in the case of the Sutterby model, the Ellis model too is a three-parameter model and has the advantage of exhibiting a limiting viscosity (i.e.,  $\mu_0$ ) in the limit of zero shear rate and shear-thinning viscosity at higher shear rates.

### 1.2.1.4 Bingham Fluid

The Bingham fluid model is characterized by a yield stress  $\tau_Y$  which, if exceeded, makes the material flow like a viscous Newtonian fluid or else behave like a solid at all values below critical stress. Hence, the equations for this fluid are as follows:

$$\tau = \tau_Y + \mu_p \dot{\gamma}, \quad |\tau| > \tau_Y \quad (1.7a)$$

$$\dot{\gamma} = 0, \quad |\tau| \leq \tau_Y \quad (1.7b)$$

Here,  $\mu_p$  is called plastic viscosity.

### 1.2.1.5 Herschel–Bulkley Fluid

This fluid model was developed by Herschel and Bulkley (1926) specifically for describing pseudoplastic fluids with yield stress and is given as follows:

$$\tau = \tau_Y + K \dot{\gamma}^n, \quad |\tau| > \tau_Y \quad (1.8a)$$

$$\dot{\gamma} = 0, \quad |\tau| \leq \tau_Y \quad (1.8b)$$

It can be seen that when yield stress is absent, this model is akin to the Ostwald–de Waele power-law model, and when  $n = 1$ , represents the Bingham plastic model.

## 1.2.2 Non-Newtonian Viscoelastic Behavior in Laminar Flow

Viscoelastic fluids described by the following constitutive equations are chosen for use in the laminar flow heat transfer analyses without the presence of porous media that are considered later in the text.

$$\tau_{ij} = \mu(\tilde{\Pi}) B_{(1)}^{ij} + \omega(\tilde{\Pi}) B_{(1)k}^i B_{(1)}^{kj} - \lambda(\tilde{\Pi}) B_{(2)}^{ij} \quad (1.9)$$

where

$$B_{(1)}^{ij} = g^{im} v_{,m}^j + g^{jm} v_{,m}^i \quad (1.10)$$

$$B_{(n+1)}^{ij} = \frac{\delta B_{(n)}^{ij}}{\delta t} \quad (1.11)$$

and the time derivative  $\delta/\delta t$  is defined as

$$\frac{\delta B_{(n)}^{ij}}{\delta t} = \frac{\partial B_{(n)}^{ij}}{\partial t} + v^k B_{(n),k}^{ij} - v_{,m}^i B_{(n)}^{mj} - v_{,m}^j B_{(n)}^{im} \quad (1.12)$$

$\tilde{\Pi}$  represents the second invariant of  $B_{(1)}^{ij}$  and  $\mu$ ,  $\omega$ , and  $\lambda$  are functions of  $\tilde{\Pi}$  only. The use of such a constitutive equation for solution of boundary-layer flows of elastic fluids has been well described by Denn (1967) and Kale *et al.* (1975). The justification is essentially due to the fact that Equation (1.9) represents the behavior of elastic fluids exactly in viscometric flows and that for the two-dimensional boundary-layer flows the dominant terms in the rate of strain tensor are those which appear in viscometric flows. The functions  $\mu$  and  $\lambda$  may often be expressed as power functions:

$$\mu(\tilde{\Pi}) = K \left[ \frac{1}{2} \tilde{\Pi} \right]^{(n-1)/2} \quad (1.13)$$

and

$$\lambda(\tilde{\Pi}) = M \left[ \frac{1}{2} \tilde{\Pi} \right]^{(s-2)/2} \quad (1.14)$$

The form of  $\omega(\tilde{\Pi})$  is unimportant since the terms in which it appears vanishes in two-dimensional flows where it will be used in the text. Note that with  $\lambda(\tilde{\Pi}) = \omega(\tilde{\Pi}) = 0$  and with  $\mu(\tilde{\Pi})$  given by Equation (1.11), the Ostwald–de Waele power-law behavior is represented.

There are a large number of available constitutive equations for viscoelastic fluids, as outlined by Bird *et al.* (1977) and Middleman (1977). However, the one used in heat transfer in porous media is the Oldroyd model for oscillatory convection.

The constitutive equation for an Oldroyd (1950) fluid can be written as follows:

$$\tau + \lambda_{RX} \dot{\tau} = \mu_0 (\dot{\gamma} + \lambda_{RD} \ddot{\gamma}) \quad (1.15)$$

where  $\mu_0$  is the constant viscosity at low shear rate in the steady-state region, that is, when  $\dot{\tau} = \ddot{\gamma} = 0$ . The constant  $\lambda_{RX}$  is a *relaxation time*, that is, if motion suddenly stops, the shear stress will decay as  $\exp(-t/\lambda_{RX})$ .  $\lambda_{RD}$  is called the *retardation time* and reflects the decay of strain rate as  $\exp(-t/\lambda_{RD})$  when all stresses are removed. When  $\lambda_{RX}$  and  $\lambda_{RD}$  are both equal to zero, the model describes a Newtonian fluid. When only  $\lambda_{RD}$  is zero, the model reverts to the Maxwell (1867) model. Thus, when stress is removed, the shear rate in a Maxwell fluid becomes zero instantaneously; whereas in an Oldroyd fluid, it decays as  $\exp(-t/\lambda_{RD})$ . This marks the essential difference between the Oldroyd and Maxwell fluids.

It is often very difficult to separate out the effects of the viscous and elastic behavior of viscoelastic fluids. However, the present state is quite different with the introduction of highly elastic constant viscosity fluids by Boger (1977/78) and

Choplin *et al.* (1983). Attempts have since been made to determine the exclusive effect of elasticity on the heat transfer characteristics in porous media through the use of elastic Boger fluids.

### 1.2.3 Non-Newtonian Viscous Behavior in Turbulent Flow

For inelastic non-Newtonian fluids in turbulent flow in smooth tubes, Dodge and Metzner (1959) have provided a Blasius type of approximate equation for the friction factor generalized Reynolds number as follows:

$$f = \frac{\alpha}{Re_{gen}^\beta} \quad 5 \times 10^3 \leq Re_{gen} \leq 10^5 \quad (1.16)$$

where  $\alpha$  and  $\beta$  are functions of  $n$  for the case of power-law fluids and their values for varying  $n$  are presented in Table 1.2.

Following the procedure of Skelland (1967), a suitable expression for the local surface shear stress was obtained by Shenoy and Mashelkar (1978b) by proper rearranging and adapting the equations for flow over a smooth flat plate at zero incidence in a manner analogous to that used by Eckert and Jackson (1950) in the Newtonian case as

$$\tau_0 = \Omega \rho^{1-\beta} \gamma_1^\beta \delta^{-\beta n} \Lambda_1^{2-\beta(2-n)} \quad (1.17)$$

where

$$\Omega = \frac{\alpha(0.817)^{2-\beta(2-n)}}{2^{\beta n+1}} \quad (1.18)$$

and

$$\gamma_1 = 8^{n-1} K \left( \frac{3n+1}{4n} \right)^n \quad (1.19)$$

**Table 1.2** Values of  $\alpha$  and  $\beta$  for varying values of pseudoplasticity index  $n$ .

$n$	$\alpha$	$\beta$
1.0	0.0790	0.250
0.9	0.0770	0.255
0.8	0.0760	0.263
0.7	0.0752	0.270
0.6	0.0740	0.281
0.5	0.0723	0.290
0.4	0.0710	0.307
0.3	0.0683	0.325
0.2	0.0646	0.349

*Source:* Dodge and Metzner (1959). Reproduced with permission of John Wiley & Sons.



For the Newtonian case

$$n = 1, \quad \beta = 0.25, \quad \Omega = 0.02332, \quad \gamma_1 = \mu$$

$$\tau_{0N} = 0.02332\rho\Lambda_1^2\left(\frac{\mu}{\rho\delta\Lambda_1}\right)^{1/4} \quad (1.20)$$

#### 1.2.4 Mildly Elastic Drag-Reducing Behavior in Turbulent Flow

Drag-reducing fluids are known to be Newtonian in viscosity but exhibit mild elasticity characterized by a relaxation time  $\theta_\beta$ . Detailed discussions on the determination of relaxation times for drag-reducing fluids are available from Argumedo *et al.* (1978) and Cho and Hartnett (1982). For such fluids, one assumes the friction factor  $f$  to be a function of the Reynolds number  $Re$  and the Deborah number  $De$  (which is the ratio of the fluid relaxation time  $\theta_\beta$  and the characteristic process time  $v/u^{*2}$ ).

Thus, the modified form of Equation (1.2) using the appropriate characteristic time and the appropriate time scale of deformation is written as

$$De = \theta_\beta u^{*2}/v \quad (1.21)$$

where  $u^*$  is the friction velocity and  $v$  is the kinematic viscosity.

The choice of characteristic time scales for defining  $De$  has been discussed by Astarita (1965), Seyer and Metzner (1969a), and Virk (1975). Reported experimental studies on the determination of fluid relaxation times published by Seyer and Metzner (1969a) show that  $\theta_\beta$  varies as  $\dot{\gamma}^{-m}$ , where  $m$  lies between 0.5 and 1.0. The general practice is to assume that  $m$  is equal to 1 so that the Deborah number can be taken as a constant independent of shear rate knowing that  $u^{*2}/v$  is directly proportional to wall shear rate. This assumption, although not truly accurate, renders itself useful for the derivation of an approximate expression for the wall shear stress in turbulently flowing drag-reducing fluids.

Assuming Deborah number to be independent of shear rate, Seyer and Metzner (1969b) wrote an expression for friction factor for turbulent flow of drag-reducing fluids as follows:

$$(2/f)^{1/2} = A_0(1 - \xi_0)^2 \ln Re f^{1/2} + (1 - \xi_0)^2 [B_0 - A_0 \ln 2(2)^{1/2}] - 3.0 \quad (1.22)$$

where the values of  $A_0$ ,  $B_0$ , and  $\xi_0$  are given by Seyer and Metzner (1969b). A straightforward manipulation of the equation, as done by Shenoy and Mashelkar (1983), to obtain an explicit Blasius-type friction factor–Reynolds number relationship is given here:

$$f = \frac{\alpha}{Re^\beta} \quad \begin{array}{l} 5 \times 10^3 \leq Re \leq 10^5 \\ 0 \leq De \leq 10 \end{array} \quad (1.23)$$

where  $\alpha$  and  $\beta$  are functions of  $De$  for the case of drag-reducing fluids, and their values for varying  $De$  are presented in Table 1.3.

It has been suggested by Virk *et al.* (1967) that the maximum drag reduction that can be achieved in practice can be described by a unique asymptote given by

**Table 1.3** Values of  $\alpha$  and  $\beta$  for varying values of Deborah numbers  $De$ .

De	$\alpha$	$\beta$
0	0.0790	0.250
1	0.0782	0.262
2	0.0787	0.271
3	0.0741	0.278
4	0.0726	0.285
5	0.0689	0.289
6	0.0655	0.292
7	0.0662	0.301
8	0.0687	0.312
9	0.0732	0.324
10	0.0762	0.334
$\geq 20$	0.420	0.550

Source: Shenoy and Mashelkar (1983). Reproduced with permission of American Chemical Society.

$\alpha = 0.42$  and  $\beta = 0.55$  in the abovementioned equation. In Table 1.2, these values correspond to the limit  $De \geq 20$ , which has been used to denote the maximum drag reduction asymptote. As can be seen, the value of  $\alpha$  is almost constant for  $1 < De < 10$ , but jumps by about 600% for a change of  $De$  from 10 to 20. Actually, this is due to the fact that around  $De = 20$  and beyond it, the  $f$  versus  $Re$  curve has a sudden change of slope and hence it is not only the value of  $\alpha$  but also the value of  $\beta$  that undergoes a sudden change. Detailed explanation of the maximum drag reduction asymptote and its uniqueness are available in the exhaustive article of Virk (1975), which can be referred to for more details. Equation (1.23) has been used when analyzing turbulent flow in horizontal pipes by Shenoy and Mashelkar (1983), in curved tubes by Shenoy *et al.* (1980), in rotating straight tubes by Shenoy (1986b), in annular ducts by Shenoy and Shintre (1986), and in vertical tubes by Shenoy (1987).

Following the procedure of Skelland (1967), a suitable expression for the local surface shear stress was obtained by Nakayama and Shenoy (1992a) from Equation (1.23) as follows:

$$\tau_w / \rho u_c^2 = \Omega (\mu / \rho u_c \delta)^\beta \quad (1.24)$$

where

$$\Omega = \alpha (0.817)^{2-\beta} / 2^{\beta+1} \quad (1.25)$$

Note that for the Newtonian case

$$\beta = 0.25, \quad \Omega = 0.02332 \quad (1.26)$$

$$\tau_{wN} / \rho u_c^2 = 0.2332 (\mu / \rho u_c \delta)^{1/4} \quad (1.27)$$