1 Ideal Fluids

1.1 Modeling by Euler's Equations

Physical laws are mainly derived from conservation principles, such as conservation of mass, conservation of momentum, and conservation of energy.

Let us consider a fluid (gas or liquid) in motion, i.e., the flow of a fluid.[1)] Let

$$\boldsymbol{u}(x, y, z, t) = \begin{pmatrix} u_1(x, y, z, t) \\ u_2(x, y, z, t) \\ u_3(x, y, z, t) \end{pmatrix}$$

be the velocity,[2)] and denote by $\varrho = \varrho(x, y, z, t)$ the density of this fluid at point $\boldsymbol{x} = (x, y, z)$ and at time instant t.

Let us take out of the fluid at a particular instant t an arbitrary portion of volume $W(t)$ with surface $\partial W(t)$. The particles of the fluid now move, and assume that $W(t + h)$ is the volume formed at the instant $t + h$ by the same particles that formed $W(t)$ at time t.

Moreover, let $\varphi = \varphi(x, y, z, t)$ be one of the functions describing a particular state of the fluid at time t at point $\boldsymbol{x}$, such as mass per unit volume (= density), interior energy per volume, momentum per volume, etc. Hence, $\int_{W(t)} \varphi \, \mathrm{d}(x, y, z)$ gives the full amount of mass or interior energy, momentum, etc., of the volume $W(t)$ under consideration.

We would like to find the change in $\int_{W(t)} \varphi \, \mathrm{d}(x, y, z)$ with respect to time, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{W(t)} (x, y, z, t) \, \mathrm{d}(x, y, z) \ . \tag{1.1}$$

1) Flows of other materials can be included too, e.g., the flow of cars on highways, provided that the density of cars or particles is sufficiently high.

2) Note that bold letters in equations normally indicate vectors or matrices.

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We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\,\mathrm{d}(x,y,z) = \lim_{h\to 0}\frac{1}{h}\left\{\int\limits_{W(t+h)} \varphi(\tilde{\mathbf{y}},t+h)\,\mathrm{d}(y_1,y_2,y_3) - \int\limits_{W(t)} \varphi(x,y,z,t)\,\mathrm{d}(x,y,z)\right\},$$

where the change from $W(t)$ to $W(t+h)$ is obviously given by the mapping

$$\tilde{\mathbf{y}} = \boldsymbol{x} + h\cdot \boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{o}(h)$$
$$\left(\tilde{\mathbf{y}} = (y_1,y_2,y_3)^T\right).$$

The error term $\boldsymbol{o}(h)$ also depends on $\boldsymbol{x}$ but the property $\lim_{h\to 0}\frac{1}{h}\boldsymbol{o}(h) = 0$ if differentiated with respect to space, provided that these spatial derivatives are bounded.

The transformation of the integral taken over the volume $W(t+h)$ to an integral over $W(t)$ by substitution requires the integrand to be multiplied by the determinant of this mapping, i.e., by

$$\begin{vmatrix} (1+h\,\partial_x u_1) & h\,\partial_y u_1 & h\,\partial_z u_1 \\ h\,\partial_x u_2 & (1+h\,\partial_y u_2) & h\,\partial_z u_2 \\ h\,\partial_x u_3 & h\,\partial_y u_3 & (1+h\,\partial_z u_3) \end{vmatrix} + o(h)$$
$$= 1 + h\cdot\left(\partial_x u_1 + \partial_y u_2 + \partial_z u_3\right) + o(h)$$
$$= 1 + h\cdot \operatorname{div}\boldsymbol{u}(\boldsymbol{x},t) + o(h)\,.$$

Taylor expansion of $\boldsymbol{V}\varphi(\tilde{\mathbf{y}},t+h)$ around $(\boldsymbol{x},t)$ therefore leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\;\mathrm{d}(x,y,z) = \int\limits_{W(t)} \left\{\partial_t\varphi + \varphi\;\operatorname{div}\boldsymbol{u} + \langle\boldsymbol{u},\nabla\varphi\rangle\right\}\;\mathrm{d}(x,y,z)\,. \tag{1.2}$$

Here, ∇v denotes the gradient of a scalar function v, and $\langle\cdot\,,\,\cdot\rangle$ means the standard scalar product of two vectors out of $\mathbb{R}^3$.

The product rule from differentiation gives:

$$\varphi\operatorname{div}\boldsymbol{u} + \langle\boldsymbol{u},\nabla\varphi\rangle = \operatorname{div}(\varphi\cdot\boldsymbol{u})\,,$$

so that (1.2) leads to the so-called *Reynolds' transport theorem*[3)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\;\mathrm{d}(x,y,z) = \int\limits_{W(t)} \left\{\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u})\right\}\;\mathrm{d}(x,y,z)\,. \tag{1.3}$$

As already mentioned, the dynamics of fluids can be described directly by conservation principles and – as far as gases are concerned – by an additional equation of state.

3) Osborne Reynolds (1842–1912); Manchester

1. Conservation of mass: If there are no sources or losses of fluid within the subdomain of the flow under consideration, the mass remains constant.

Because $W(t)$ and $W(t+h)$ consist of the same particles, they have the same mass. The mass of $W(t)$ is given by $\int_{W(t)} \varrho(x, y, z, t)\, \mathrm{d}(x, y, z)$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{W(t)} \varrho(x, y, z, t)\, \mathrm{d}(x, y, z) = 0$$

must hold. Taking (1.3) into account (particularly for $\varphi = \varrho$), this leads to the requirement

$$\int\limits_{W(t)} \{\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u})\}\, \mathrm{d}(x, y, z) = 0\,.$$

Since this has to hold for arbitrary $W(t)$, the integrand has to vanish:

$$\boxed{\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0\,.} \tag{1.4}$$

This equation is called the *continuity equation.*

2. Conservation of momentum: Another conservation principle concerns the momentum of a mass, which is defined as

mass × velocity .

Thus,

$$\int\limits_{W(t)} \varrho \boldsymbol{u}\, \mathrm{d}(x, y, z)$$

gives the momentum of the mass at time t of the volume $W(t)$ and

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \varrho \boldsymbol{u}$$

describes the *density of momentum.*

The principle of the conservation of momentum, i.e., Newton's second law

force = mass × acceleration,

then states that the change of momentum with respect to time equals the sum of all of the exterior forces acting on the mass of $W(t)$.

In order to describe these exterior forces, we take into account that there is a certain pressure $p(\boldsymbol{x}, t)$ at each point $\boldsymbol{x}$ in the fluid at each instant t. If $\boldsymbol{n}$ is considered to be the unit vector normal on the surface $\partial W(t)$ of $W(t)$, and it is directed outwards, the fluid outside of $W(t)$ acts on $W(t)$ with a force given by

$$-\int_{\partial W(t)} p\,\boldsymbol{n}\,\mathrm{do} \quad (\mathrm{do} = \text{area element of } \partial W(t))\ .$$

Besides the normal forces per unit surface area generated by the pressure, there are also tangential forces which act on the surface due to the friction generated by exterior particles along the surface.

Though this so-called fluid *viscosity* leads to a lot of remarkable phenomena, we are going to neglect this property at the first step. Instead of *real fluids* or *viscous fluids*, we restrict ourselves in this chapter to so-called *ideal fluids* or *inviscid fluids*. This restriction to ideal fluids, particularly to *ideal gases*, is one of the idealizations mentioned in the Preface.

However, as well as exterior forces per unit surface area, there are also exterior forces per unit volume – e.g., the weight.

Let us denote these forces per unit volume by $\boldsymbol{k}$, such that Newton's second law leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W(t)} \boldsymbol{q}\,\mathrm{d}(x,y,z) = \int_{W(t)} \boldsymbol{k}(x,y,z,t)\,\mathrm{d}(x,y,z) - \int_{\partial W(t)} p\cdot\boldsymbol{n}\,\mathrm{do}\ .$$

Thus, by Gauss' divergence theorem, we find

$$\int_{\partial W(t)} p\,\boldsymbol{n}\,\mathrm{do} = \begin{pmatrix} \int_{\partial W(t)} p\,n_1\,\mathrm{do} \\ \int_{\partial W(t)} p\,n_2\,\mathrm{do} \\ \int_{\partial W(t)} p\,n_3\,\mathrm{do} \end{pmatrix} = \begin{pmatrix} \int_{W(t)} \partial_x p\,\mathrm{d}(x,y,z) \\ \int_{W(t)} \partial_y p\,\mathrm{d}(x,y,z) \\ \int_{W(t)} \partial_z p\,\mathrm{d}(x,y,z) \end{pmatrix}$$

$$= \int_{W(t)} \nabla p\,\mathrm{d}(x,y,z)\ .$$

Together with (1.3),

$$\int_{W(t)} \left\{ \partial_t\boldsymbol{q} + \begin{pmatrix} \operatorname{div}(q_1\boldsymbol{u}) \\ \operatorname{div}(q_2\boldsymbol{u}) \\ \operatorname{div}(q_3\boldsymbol{u}) \end{pmatrix} - \boldsymbol{k} + \nabla p \right\} \mathrm{d}(x,y,z) = 0$$

follows.

Again, this has to be valid for any arbitrarily chosen volume $W(t)$. If, moreover,

$$\operatorname{div}(q_i\,\boldsymbol{u}) = \langle \boldsymbol{u}, \nabla q_i\rangle + \operatorname{div}\boldsymbol{u}\cdot q_i$$

is taken into account,

$$\partial_t\boldsymbol{q} + \langle \boldsymbol{u}, \nabla\rangle\ \boldsymbol{q} + \operatorname{div}\boldsymbol{u}\cdot\boldsymbol{q} + \nabla p = \boldsymbol{k}\ ,$$

i.e.,

$$\boxed{\partial_t \boldsymbol{q} + \frac{1}{\varrho}\,\langle \boldsymbol{q}, \nabla\rangle\ \boldsymbol{q} + \operatorname{div}\left(\frac{1}{\varrho}\boldsymbol{q}\right)\ \boldsymbol{q} + \nabla p = \boldsymbol{k}} \tag{1.5}$$

has to be fulfilled.

The number of equations represented by (1.5) equals the spatial dimension of the flow, i.e., the number of components of $\boldsymbol{q}$ or $\boldsymbol{u}$.

By means of the continuity equation, (1.5) can be reformulated as

$$\boxed{\partial_t \boldsymbol{u} + \langle \boldsymbol{u}, \nabla\rangle\ \boldsymbol{u} + \frac{1}{\varrho}\,\nabla p = \hat{\boldsymbol{k}}\ ,} \tag{1.6}$$

where the force $\boldsymbol{k}$ per unit volume has been replaced by the force $\hat{\boldsymbol{k}} = \frac{1}{\varrho}\,\boldsymbol{k}$ per unit mass.

Equation (1.4) together with (1.5) or (1.6) are called *Euler's equations.*[4)]

$$\varrho,\ E,\ q_1,\ q_2,\ q_3$$

are sometimes called *conservative variables* whereas $\varrho,\ \varepsilon,\ u_1,\ u_2,\ u_3$ are the *primitive variables*. Here,

$$E := \varrho\,\varepsilon + \frac{\varrho}{2}\,\|\boldsymbol{u}\|^2 = \varrho\,\varepsilon + \frac{\|\boldsymbol{q}\|^2}{2\varrho}$$

gives the *total energy* per unit volume, where ε stands for the *interior energy* per unit mass, e.g., the heat per unit mass.

$$\frac{\varrho}{2}\,\|\boldsymbol{u}\|^2$$

obviously introduces the *kinetic energy* per unit volume.[5)]

$\langle \boldsymbol{u}, \nabla\rangle\ \boldsymbol{u}$ is called the *convection term.*

Remark

Terms of the form $\partial_t \boldsymbol{w} + \langle \boldsymbol{w}, \nabla\rangle\ \boldsymbol{w}$ are often abbreviated in the literature to $\frac{D\boldsymbol{w}}{Dt}$ and are called *material time derivatives* of the vector-valued function $\boldsymbol{w}$.

3. Conservation of energy: Next we consider the *first law of thermodynamics*, namely:

The change per time unit in the total energy of the mass of a moving fluid volume equals the work done per time unit against the exterior forces.

4) Leonhard Euler (1707–1783); Basel, Berlin, St. Petersburg

5) $\|\cdot\|$: 2-norm

For ideal fluids, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W(t)} E(x,y,z,t)\,\mathrm{d}(x,y,z) = \int_{W(t)} \left\langle \varrho\,\hat{\boldsymbol{k}}, \boldsymbol{u}\right\rangle \,\mathrm{d}(x,y,z) - \int_{\partial W(t)} \langle p\,\boldsymbol{n}, \boldsymbol{u}\rangle \,\mathrm{do}\,.^{6)}$$

The relation

$$\int_{\partial W(t)} \langle p\,\boldsymbol{n}, \boldsymbol{u}\rangle \,\mathrm{do} = \int_{\partial W(t)} \langle p\,\boldsymbol{u}, \boldsymbol{n}\rangle \,\mathrm{do} = \int_{W(t)} \mathrm{div}(p\,\boldsymbol{u})\,d(x,y,z)$$

follows from Gauss' divergence theorem, so that (1.3) leads to

$$\int_{W(t)} \left\{\partial_t E + \mathrm{div}(E\cdot\boldsymbol{u}) + \mathrm{div}(p\cdot\,\boldsymbol{u}) - \left\langle\varrho\,\hat{\boldsymbol{k}}, \boldsymbol{u}\right\rangle\right\}\,\mathrm{d}(x,y,z) = 0\,, \quad \forall\, W(t)\,.$$

If $\boldsymbol{k}$ can be neglected because of the small weight of the gas, or if $\hat{\boldsymbol{k}}$ is the weight of the fluid per unit mass[7] and $\boldsymbol{u}$ is the velocity of flow parallel to the Earth's surface, we get

$$\boxed{\partial_t E + \mathrm{div}\left(\frac{E+p}{\varrho}\,\boldsymbol{q}\right) = 0\,.} \tag{1.7}$$

Explicitly written, and neglecting $\boldsymbol{k}$, Eqs. (1.4), (1.5) and (1.7) become

$$\begin{aligned}
&\partial_t\varrho + \partial_x q_1 + \partial_y q_2 + \partial_z q_3 = 0\\
&\partial_t q_1 + \partial_x\left(\frac{1}{\varrho}\,q_1q_1 + p\right) + \partial_y\left(\frac{1}{\varrho}\,q_1q_2\right) + \partial_z\left(\frac{1}{\varrho}\,q_1q_3\right) = 0\\
&\partial_t q_2 + \partial_x\left(\frac{1}{\varrho}\,q_2q_1\right) + \partial_y\left(\frac{1}{\varrho}\,q_2q_2 + p\right) + \partial_z\left(\frac{1}{\varrho}\,q_2q_3\right) = 0\\
&\partial_t q_3 + \partial_x\left(\frac{1}{\varrho}\,q_3q_1\right) + \partial_y\left(\frac{1}{\varrho}\,q_3q_2\right) + \partial_z\left(\frac{1}{\varrho}\,q_3q_3 + p\right) = 0\\
&\partial_t E + \partial_x\left(\frac{E+p}{\varrho}\,q_1\right) + \partial_y\left(\frac{E+p}{\varrho}\,q_2\right) + \partial_z\left(\frac{E+p}{\varrho}\,q_3\right) = 0\,.
\end{aligned} \tag{1.8}$$

Hence, we are concerned with a system of equations that can be used to determine the functions ϱ, q_1, q_2, q_3, E. However, we must note that there is an additional function that is sought, namely the pressure p.

In the case of constant density[8] ϱ, i.e., $\partial_t\varrho = 0\ \ \forall\,(x,y,z)$, only four conservation variables have to be determined, so the five equations in (1.8) are sufficient. Otherwise, particularly in the case of gas flow, a sixth equation is needed, namely an

6) Work = force × length = pressure × area × length ⇒ $\frac{\text{work}}{\text{time}}$ = force × velocity = pressure × area × velocity

7) In other words, $\|\hat{\boldsymbol{k}}\| = g$ where g is the Earth's gravitational acceleration.

8) The case of constant density is not necessarily identical to the case of an *incompressible* flow defined by $\mathrm{div}\,\boldsymbol{u} = 0$, because in this case the continuity equation (1.4) is already fulfilled if the pair $(\varrho\boldsymbol{u})$ shows the property $\partial_t\varrho + \langle\boldsymbol{u}\nabla\varrho\rangle = 0$, which does not necessarily imply ϱ = const.

equation of state. State variables of a gas are:

$$T = \text{temperature}\ , \quad p = \text{pressure}\ , \quad \varrho = \text{density}\ , \quad V = \text{volume}\ ,$$
$$\varepsilon = \text{energy/mass}\ , \quad S = \text{entropy/mass}\ ,$$

and a theorem of thermodynamics says that each of these state variables can be uniquely expressed in terms of two of the other state variables. Such relations between three state variables are called *equations of state*. Thus, p can be expressed by ϱ and ε (and hence by $\varrho, E, \boldsymbol{q}$); for inviscid so-called γ*-gases*, the relation is given by:

$$p = (\gamma - 1)\,\varrho\varepsilon = (\gamma - 1)\,\left(E - \frac{\|\boldsymbol{q}\|^2}{2\varrho}\right) \tag{1.9}$$

with $\gamma = \text{const} > 1$, such that only the functions ϱ, $\boldsymbol{q}$, E have to be determined. Here, the *adiabatic exponent* γ is the ratio $\frac{c_p}{c_V}$ of the specific heats. In the case of air, we have $\gamma \approx 1.4$.

Using vector-valued functions, systems of differential equations can also be described by a single differential equation. In this way, and taking (1.9) into account as an additional equation, (1.8) can be written as

$$\partial_t \boldsymbol{V} + \partial_x \boldsymbol{f}_1(\boldsymbol{V}) + \partial_y \boldsymbol{f}_2(\boldsymbol{V}) + \partial_z \boldsymbol{f}_3(\boldsymbol{V}) = 0 \tag{1.10}$$

with

$$\boldsymbol{V} = (\varrho, q_1, q_2, q_3, E)^T$$

and with

$$\boldsymbol{f}_1(\boldsymbol{V}) = \begin{pmatrix} q_1 \\ \frac{1}{\varrho}\, q_1 q_1 + p \\ \frac{1}{\varrho}\, q_2 q_1 \\ \frac{1}{\varrho}\, q_3 q_1 \\ \frac{E+p}{\varrho}\, q_1 \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{V}) = \begin{pmatrix} q_2 \\ \frac{1}{\varrho}\, q_2 q_1 \\ \frac{1}{\varrho}\, q_2 q_2 + p \\ \frac{1}{\varrho}\, q_3 q_2 \\ \frac{E+p}{\varrho}\, q_2 \end{pmatrix}, \quad \boldsymbol{f}_3(\boldsymbol{V}) = \begin{pmatrix} q_3 \\ \frac{1}{\varrho}\, q_3 q_1 \\ \frac{1}{\varrho}\, q_3 q_2 \\ \frac{1}{\varrho}\, q_3 q_3 + p \\ \frac{E+p}{\varrho}\, q_3 \end{pmatrix}.$$

The functions $\boldsymbol{f}_j(\boldsymbol{V})$ are called *fluxes*.

If $\boldsymbol{Jf}_1, \boldsymbol{Jf}_2, \boldsymbol{Jf}_3$ are the *Jacobians*, (1.10) becomes

$$\partial_t \boldsymbol{V} + \boldsymbol{Jf}_1(\boldsymbol{V}) \cdot \partial_x \boldsymbol{V} + \boldsymbol{Jf}_2(\boldsymbol{V}) \cdot \partial_y \boldsymbol{V} + \boldsymbol{Jf}_3(\boldsymbol{V}) \cdot \partial_z \boldsymbol{V} = 0\ . \tag{1.11}$$

Because of their particular meaning in physics, systems of differential equations of type (1.10) are called systems of *conservation laws*, even if they do not arise from physical aspects. Obviously, (1.11) is a quasilinear system of first-order partial differential equations.

Such systems must be defined by initial conditions

$$\boldsymbol{V}(\boldsymbol{x}, 0) = \boldsymbol{V}_0(\boldsymbol{x})\ , \tag{1.12}$$

where $\boldsymbol{V}_0(\boldsymbol{x})$ is a prescribed initial state, and by boundary conditions.

If the flow does not depend on time, it is called a *stationary* or steady-state flow, and initial conditions do not occur. Otherwise, the flow is termed *nonstationary*.

As far as ideal fluids are concerned, it can be assumed that the fluid flow is tangential along the surface of a solid body[9] fixed in space or along the bank of a river, etc. In this case,

$$\langle \boldsymbol{u}, \boldsymbol{n} \rangle = 0 \tag{1.13}$$

is one of the boundary conditions, where $\boldsymbol{n}$ are the outward-directed normal unit vectors along the surface of the body.

There are often symmetries with respect to space such that the number of unknowns can be reduced, e.g., in the case of rotational symmetry combined with polar coordinates. If it is found that there is only one spatial coordinate, the problem is termed *one-dimensional*.[10] If rectangular space variables are used and x is the only one that remains, we end up with the system

$$\partial_t \boldsymbol{V} + \boldsymbol{J f}(\boldsymbol{V}) \cdot \partial_x \boldsymbol{V} = 0 \tag{1.14}$$

with

$$\boldsymbol{V} = \begin{pmatrix} \varrho \\ q \\ E \end{pmatrix} , \quad \boldsymbol{f}(\boldsymbol{V}) = \begin{pmatrix} q \\ \dfrac{1}{\varrho} q^2 + p \\ \dfrac{E+p}{\varrho} q \end{pmatrix}$$

and with the equation of state

$$p = (\gamma - 1) \left(E - \frac{q^2}{2\varrho} \right) .$$

Hence,

$$\boldsymbol{J f}(\boldsymbol{V}) = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{3-\gamma}{2} \dfrac{q^2}{\varrho^2} & (3-\gamma) \dfrac{q}{\varrho} & \gamma - 1 \\ (\gamma-1) \dfrac{q^3}{\varrho^3} - \gamma \dfrac{E q}{\varrho^2} & \gamma \dfrac{E}{\varrho} - \dfrac{3(\gamma-1)}{2} \dfrac{q^2}{\varrho^2} & \gamma \dfrac{q}{\varrho} \end{pmatrix} .$$

As can easily be verified, $\lambda_1 = \frac{q}{\varrho}$ is a solution of the characteristic equation of $\boldsymbol{J f}(\boldsymbol{V})$, namely of

$$\begin{aligned} &-\lambda^3 + 3 \frac{q}{\varrho} \lambda^2 - \left\{ (\gamma^2 - \gamma + 6) \frac{q^2}{2\varrho^2} - \gamma(\gamma-1) \frac{E}{\varrho} \right\} \lambda \\ &+ \left(\frac{\gamma^2}{2} - \frac{\gamma}{2} + 1 \right) \frac{q^3}{\varrho^3} - \gamma(\gamma-1) \frac{E q}{\varrho^2} = 0 . \end{aligned}$$

9) For example, a wing

10) Two- or three-dimensional problems are defined analogously.

The other two eigenvalues of $Jf(V)$ are then the roots of

$$\lambda^2 - \frac{2q}{\varrho}\lambda + (\gamma^2 - \gamma + 2)\frac{q^2}{2\varrho^2} - \gamma(\gamma - 1)\frac{E}{\varrho} = 0:$$

$$\begin{aligned}\lambda_{2,3} &= \frac{q}{\varrho} \pm \sqrt{\frac{q^2}{\varrho^2} + \gamma(\gamma-1)\frac{E}{\varrho} - (\gamma^2-\gamma+2)\frac{q^2}{2\varrho^2}} \\ &= \frac{q}{\varrho} \pm \sqrt{\gamma(\gamma-1)\frac{E}{\varrho} - \gamma(\gamma-1)\frac{q^2}{2\varrho^2}} \\ &= \frac{q}{\varrho} \pm \sqrt{\frac{\gamma(\gamma-1)}{\varrho}\left[E - \frac{q^2}{2\varrho}\right]} = \frac{q}{\varrho} \pm \sqrt{\gamma\frac{p}{\varrho}}\,.\end{aligned} \tag{1.15}$$

Thus, these eigenvalues are real and different from each other ($p > 0$).

Definition

If all the eigenvalues of a matrix $A(x, t, V)$ are real and if the matrix can be diagonalized, the system of equations

$$\partial_t V + A(x, t, V)\,\partial_x V = 0$$

is termed *hyperbolic* at (x, t, V). If the eigenvalues are real and different from each other such that A can definitely be diagonalized, the system is said to be *strictly hyperbolic*.

Obviously then, (1.14) is strictly hyperbolic for all (x, t, V) under consideration.

By the way, because $\frac{q}{\varrho} = u$, the velocity of the flow, and because $\sqrt{\frac{\gamma p}{\varrho}}$ describes the local sound velocity $\hat{c}$ (cf. (1.66)), the eigenvalues are:

$$\lambda_1 = u\,, \quad \lambda_2 = u + \hat{c}\,, \quad \lambda_3 = u - \hat{c}\,,$$

and have equal signs in the case of supersonic flow, whereas the subsonic flow is characterized by the fact that one eigenvalue has a different sign to the others.

Definition

Let $u(x, t_0)$ be the velocity field of the flow at instant t_0 and let x_0 be an arbitrary point from the particular subset of $\mathbb{R}^3$ which is occupied by the fluid at this particular instant. If, at this moment, the system

$$\left[x'(s), u(x(s), t_0)\right] = 0^{11)}$$

of ordinary differential equations with the initial condition

$$x(0) = x_0$$

has a unique solution $x = x(s)$ for each of these points x_0, each of the curves $x = x(s)$ is called a *streamline* at instant t_0.

Here, the streamlines may be parametrized by the arc length s with $s = 0$ at point $\boldsymbol{x}_0$, such that

$$\langle \boldsymbol{x}', \boldsymbol{x}' \rangle = 1 \, .$$

Thus, the set of streamlines at an instant t_0 shows a snapshot of the flow at this particular instant. It does not necessarily describe the trajectories along which the fluid particles move over time. Only if the flow is a stationary one, i.e., if $\boldsymbol{u}$, ϱ, p, $\hat{\boldsymbol{k}}$ are independent of time, do the streamlines and trajectories coincide: a particle moves along a fixed streamline over time.

1.2 Characteristics and Singularities

As an introductory example for a more general investigation of conservation laws, let us consider the scalar *Burgers' equation*[12] without exterior forces:

$$\partial_t v + \partial_x \left(\frac{1}{2} v^2 \right) = 0 \, , \quad (x,t) \in \Omega \, , \quad \text{i.e.,} \quad x \in \mathbb{R} \, , \quad t \geq 0 \, , \tag{1.16}$$

which is often studied as a model problem from a theoretical point of view and also as a test problem for numerical procedures.

Here, the flux is given by $f(v) = \frac{1}{2}v^2$. If

$$v_0(x) = 1 - \frac{x}{2} \tag{1.16a}$$

is chosen as a particular example of an initial condition, the *unique* and *smooth solution*[13] turns out to be

$$v(x,t) = \frac{2-x}{2-t} \, ,$$

but this solution only exists locally, namely for $0 \leq t < 2$; as time increases it runs into a singularity at $t = 2$.

As a matter of fact, classical existence and uniqueness theorems for quasilinear first-order partial differential equations with smooth coefficients only ensure the unique existence of a classical smooth solution in a certain neighborhood of the initial manifold, provided that the initial condition is also sufficiently smooth.

The occurrence of discontinuities or singularities does not depend on the smoothness of the fluxes: assume that $v(x,t)$ is a smooth solution of the problem

$$\partial_t v + \partial_x f(v) = 0 \quad \text{for} \quad x \in \mathbb{R} \, , \quad t \geq 0$$
$$v(x,0) = v_0(x)$$

11) Here, $[\cdot,\cdot]$ is the vector product in $\mathbb{R}^3$.

12) J. Burgers: Nederl. Akad. van Wetenschappen **43** (1940) 2–12.

13) In other words, v is continuously differentiable.

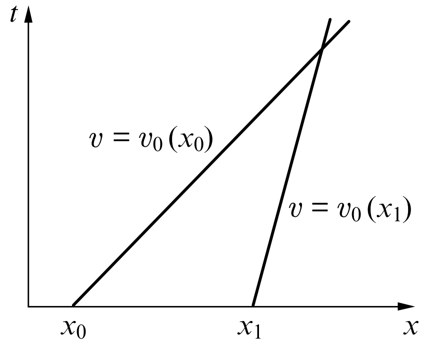

Fig. 1.1 Formation of discontinuities.

in a certain neighborhood immediately above the x-axis. Obviously, this solution is constant along the straight line

$$x(t) = x_0 + t f'(v_0(x_0)) \tag{1.17}$$

that crosses the x-axis at x_0, where $x_0 \in \mathbb{R}$ is chosen arbitrarily. Its value along this line is therefore $v(x(t), t) = v_0(x_0)$. This can easily be verified by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v(x(t), t) &= \partial_t v(x(t), t) + \partial_x v(x(t), t) \cdot x'(t) \\ &= \partial_t v(x, t) + \partial_x v(x, t) \cdot f'(v_0(x_0)) \\ &= \partial_t v(x, t) + \partial_x v(x, t) \cdot f'(v(x, t)) = \partial_t v + \partial_x f(v) = 0 \,. \end{aligned}$$

The straight lines of (1.17), each of which belong to a particular x_0, are called the *characteristics* of the given conservation law.

In the case of $x_0 < x_1$, but for example $0 < f'(v_0(x_1)) < f'(v_0(x_0))$,[14)] the characteristic through $(x_0, 0)$ intersects the characteristic through $(x_1, 0)$ at an instant $t_1 > 0$, so that at the point of intersection the solution v must have the value $v_0(x_0)$ as well as the value $v_0(x_1) \neq v_0(x_0)$. Therefore, a discontinuity will occur at the instant t_1 or even earlier. With respect to fluid dynamics, one notable type of discontinuity is shocks (discussed later).

When applied to Burgers' equation (1.16) with an initial function of (1.16a), the characteristic through a point x_0 on the x-axis is given by

$$t = 2\,\frac{x - x_0}{2 - x_0} \,,$$

such that the discontinuity we found at $t = 2$ can also be immediately understood via Fig. 1.2.

If systems of conservation laws are considered instead of the scalar case, i.e.,

$$V(x, t) \in \mathbb{R}^m \,, \; m \in \mathbb{N} \,, \; \forall\,(x, t) \in \Omega \,,$$

and if only one spatial variable x occurs, a system of characteristics is defined as follows:

14) Hence, $v_0(x_0) \neq v_0(x_1)$ too

Definition

If $V(x, t)$ is a solution of (1.10), in the case of only one space variable x, the one-parameter set

$$x_{(i)} = x_{(i)}(t, \chi) \quad (\chi \in \mathbb{R} : \text{set parameter})$$

of real curves defined by the ordinary differential equation

$$\dot{x} = \lambda_i V(x, t) \tag{1.18}$$

for every fixed i ($i = 1, \ldots, m$) is called the set of *i-characteristics* of the particular system that belongs to V. Here, $\lambda_i V)$ ($i = 1, \cdots, m$) are the eigenvalues of the Jacobian $\boldsymbol{Jf}(\boldsymbol{V})$.

Obviously, this definition coincides in the case of $m = 1$ with the previously presented definition of characteristics.

Let us finally – using an example – study the situation for a system of conservation laws if the system is linear with constant coefficients:

$$\partial_t \boldsymbol{V} + \boldsymbol{A}\,\partial_x \boldsymbol{V} = 0 \tag{1.19}$$

with a constant Jacobian (m, m)-matrix $\boldsymbol{A}$. Moreover, let us assume the system to be strictly hyperbolic.

From (1.18), the characteristics turn out to be the set of straight lines given by

$$x_{(i)}(t) = \lambda_i\, t + x_{(i)}(0)\ , \quad (i = 1, \ldots, m)\ ,$$

and are independent of $\boldsymbol{V}$. Hence, for every fixed i, the characteristics belonging to this set are parallel.

If $\boldsymbol{S} = (s_1, s_2, \ldots, s_m)$ is the matrix whose columns consist of the eigenvectors of the Jacobian, and if $\boldsymbol{\Lambda}$ denotes the diagonal matrix consisting of the eigenvalues of $\boldsymbol{A}$,

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\,\boldsymbol{S}^{-1} \tag{1.20}$$

follows.

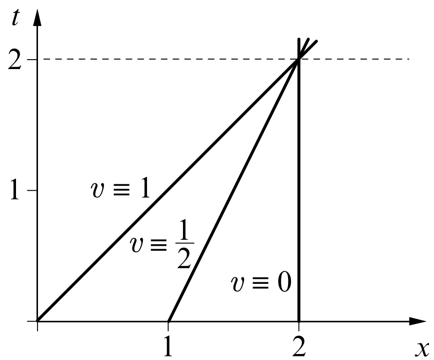

Fig. 1.2 Discontinuity of the solution to Burgers' equation.

If new variables $\hat{V}$ are introduced by

$$\hat{V} = S^{-1} V ,$$

the system takes the form

$$\partial_t \hat{V} + \Lambda\, \partial_x \hat{V} = 0 , \quad \hat{V}_0 = S^{-1} V_0 . \tag{1.21}$$

This is a decoupled system:

$$\partial_t \hat{v}_i + \lambda_i\, \partial_x \hat{v}_i = 0 , \;\; \hat{v}_i(x, 0) = \left[S^{-1} V_0(x) \right]_i = \hat{v}_{i_0}(x) , \quad (i = 1, \ldots , m) .$$

Each of the equations of this system is an independent scalar equation called an *advection equation*. The solution is

$$\hat{v}_i(x, t) = \hat{v}_{i_0}(x - \lambda_i\, t) \quad (i = 1, \ldots , m) . \tag{1.22}$$

Hence, the state at instant t moves with velocity λ_i in the positive or negative x-direction according to the sign of λ_i. This is called wave propagation, where the velocity of propagation is described by λ_i.

Obviously,

$$V = (s_1, \ldots , s_m) \begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_m \end{pmatrix} = \hat{v}_1 s_1 + \hat{v}_2 s_2 + \ldots + \hat{v}_m\, s_m .$$

Each of the vector-valued functions

$$V_{(i)}(x, t) := \hat{v}_i(x, t) \cdot s_i$$

solves the system of differential equations because of

$$\begin{aligned} \partial_t V_{(i)} + S \Lambda S^{-1}\, \partial_x V_{(i)} &= \partial_t \hat{v}_i\, s_i + \partial_x \hat{v}_i\, \lambda_i\, s_i \\ &= (\partial_t \hat{v}_i + \lambda_i\, \partial_x \hat{v}_i)\; s_i = 0 . \end{aligned}$$

$V_{(i)}$ $(i = 1, \ldots , m)$ is often called the *solution belonging to the i-th set of characteristics and to the given initial value* $\hat{v}_{i_0}(x)$. Obviously, the vector functions $V_{(i)}$ $(i = 1, \ldots , m)$ are linearly independent.

Remark

The fact that a sufficiently smooth solution of a nonlinear initial value problem often only exists in the neighborhood of the initial manifold, while the corresponding real-world process exists globally, means that this solution is not accepted by the physicist or engineer. The mathematician is asked to find a global solution. This forces the mathematician to create a more general definition of the solution such that the real-world situation can be described in a satisfactory way. Suitable definitions of weak solutions will be presented in Chapter 2.

Remark

As far as the scalar linear problem

$$\partial_t v + \partial_x(a\,v) = 0\ , \quad a = \text{const}\ , \tag{1.23}$$

is concerned, v often describes a concentration, and the flux is then simply given by $f(v) = a\,v$.

Many physical processes include a further flux of the particular form

$$-\varepsilon\,\partial_x v \quad (\varepsilon > 0)\ ,$$

proportional to the drop in concentration. The transport phenomenon is then mathematically modeled by

$$\partial_t v + \partial_x\,(a\,v - \varepsilon\,\partial_x v) = 0\ , \quad \text{i.e.,} \quad \partial_t v + a\,\partial_x v = \varepsilon\,\partial_{xx} v\ . \tag{1.24}$$

Because of the *diffusion term* on the right hand side, this equation is of parabolic type. Parabolic equations yield smoother and smoother solutions as time t increases. Therefore, shocks will be smeared out as soon as diffusion occurs.

This effect of parabolic equations can easily be demonstrated using examples like the following one. The sum

$$v(x,t) = e^{\frac{a}{2}\left(x - \frac{a}{2}t\right)} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \frac{\sin(\nu x)}{\nu} e^{-\nu^2 t}$$

converges uniformly for $t > 0$ because of the factors $e^{-\nu^2 t}$. This also holds after multiple termwise differentiations of this sum with respect to t as well as with respect to x so that v is a sufficiently smooth function for all $x \in \mathbb{R},\ t > 0$. It solves (1.24)[15] according to the classical understanding, but leads for $t = 0$ to the function

$$v_0(x) = e^{\frac{a}{2}x} \cdot w(x)$$

where $w(x)$ represents the Fourier expansion of the 2π-periodic discontinuous function

$$w(x) = \begin{cases} \frac{x}{2} & \text{for} \quad -\pi < x < \pi \\ 0 & \text{for} \quad x = \pm\pi \end{cases} .$$

The curve described by $w(x)$ is sometimes called a saw blade curve.

1.3 Potential Flows and (Dynamic) Buoyancy

Let us now try to investigate the forces acting on solid bodies[16] dipped into a fluid flow at a fixed position. For convenience, we restrict ourselves to stationary flows of

15) In the case of $\varepsilon = 1$

16) For example, on the wings of an aircraft

inviscid fluids. Moreover, it will be assumed that the magnitudes of the velocities are such that the density can be regarded as a constant. Thus, the flow is incompressible and the partial derivatives with respect to t occurring in (1.8) vanish. The first equation in (1.8) – if written in primitive variables, i.e., the continuity equation (1.4) – then reduces to

$$\operatorname{div} \boldsymbol{u} = 0 \,, \tag{1.25}$$

where[17] $\boldsymbol{u} = \left(u_1(x, y, z)\,,\ u_2(x, y, z)\,,\ u_3(x, y, z)\right)^T$ again denotes the velocity vector of the flow at the space position $\boldsymbol{x} = (x, y, z)^T$.

The second, third, and fourth equations of (1.8) formulated by means of primitive variables could be written as the vector-valued Euler equation (1.6), and they lead in the case of a stationary flow to

$$\langle \boldsymbol{u}, \nabla \rangle\ \boldsymbol{u} + \frac{1}{\varrho}\, \nabla p = \hat{\boldsymbol{k}} \,. \tag{1.26}$$

Let us also assume that the flow is *irrotational*; this means that the *circulation*

$$Z := \oint_C \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{1.27}$$

vanishes for every closed contour C within every simply connected subdomain of the flow area. Then $\boldsymbol{u}$ can be derived from a potential ϕ; i.e., there is a scalar function ϕ such that

$$\boldsymbol{u} = \nabla \phi \,, \tag{1.28}$$

and

$$\operatorname{curl} \boldsymbol{u} = 0 \tag{1.29}$$

holds within this area.

By the way, the vector $\operatorname{curl} \boldsymbol{u}$ is often called the *vorticity vector* or *angular velocity vector*, and a trajectory of a field of vorticity vectors is called a *vortex line*.

Because of (1.28), a flow of this type is called a *potential flow*, and ϕ is the so-called *velocity potential*.

The fifth equation in (1.8) can be omitted assuming that knowledge of the energy density E is of no interest.

From (1.29), the relation $\langle \boldsymbol{u}, \nabla \rangle\, \boldsymbol{u} = \frac{1}{2}\nabla(\|\boldsymbol{u}\|^2) - [\boldsymbol{u}, \operatorname{curl} \boldsymbol{u}]$ leads together with (1.26) to

$$\nabla \left(\frac{1}{2}\, \|\boldsymbol{u}\|^2 + \frac{1}{\varrho}\, p \right) = \hat{\boldsymbol{k}} \,. \tag{1.30}$$

17) For the time being, z denotes the third space variable; it will later denote the complex variable $x + i y$, but we have taken care to avoid any confusion.

Formula (1.30) shows that a fluid flow of the particular type under consideration, namely an approximately stationary, inviscid, incompressible, and irrotational flow can only exist if the exterior forces are the gradient of a scalar function. In other words, these forces must be conservative, i.e., there must be a potential Q with $\hat{\boldsymbol{k}} = -\nabla Q$ such that (1.30) leads to

$$\nabla \left(\frac{1}{2} \|\boldsymbol{u}\|^2 + \frac{1}{\varrho} p + Q \right) = 0 .$$

i.e., to

$$\frac{1}{2} \|\boldsymbol{u}\|^2 + \frac{1}{\varrho} p + Q = \text{const} . \tag{1.31}$$

Equation (1.31) is called the *Bernoulli equation*,[18] and is none other than the energy conservation law for this particular type of flow. p is called the *static pressure*, whereas the term $\frac{\varrho}{2} \|\boldsymbol{u}\|^2$, i.e., the kinetic energy per volume, is often called the *dynamic pressure*.

As far as incompressible flow in a circular pipe is concerned, the velocity will necessarily increase as soon as the diameter of the pipe decreases, and – because of (1.31) – this will lead to decreasing pressure within the narrow part of the pipe. This phenomenon is called the *hydrodynamic paradox*. Applications include carburetors and jet streams.

Remark

A necessary condition for the existence of irrotational flow was the conservative character of the exterior forces. Let us now assume that these forces are conservative instead, i.e., $\hat{\boldsymbol{k}} = -\nabla Q$. Moreover, let us allow the flow to be compressible, with a particular dependence $\varrho = \varrho(p)$, termed *barotropic flow*. Integration of (1.26) along a streamline from a constant point P_0 to a variable point P then leads to

$$\int_{P_0}^{P} \left\{ \frac{1}{2} \nabla(\|\boldsymbol{u}\|^2) + [\operatorname{curl} \boldsymbol{u}, \boldsymbol{u}] + \nabla \Theta + \nabla Q \right\} \, \mathrm{d}\boldsymbol{s} = 0 ,$$

with

$$\Theta := \int \frac{\mathrm{d}p}{\varrho(p)} ,$$

hence

$$\nabla \Theta = \frac{1}{\varrho} \nabla p ,$$

18) Daniel Bernoulli (1700–1782); St. Petersburg, Basel

and with

$$\mathrm{d}\boldsymbol{s} = \frac{1}{\|\boldsymbol{u}\|}\,\boldsymbol{u}\,\mathrm{d}s \quad (s = \text{arc length})\ .$$

Because P was arbitrary, and because of $\langle[\operatorname{curl}\boldsymbol{u},\boldsymbol{u}],\boldsymbol{u}\rangle = 0$, this result leads to the *generalized Bernoulli equation*

$$\frac{1}{2}\,\|\boldsymbol{u}\|^2 + \Theta + Q = \text{const}\ ,$$

which in the case of constant density, i.e., $\Theta = \frac{p}{\varrho}$, seems to coincide completely with (1.31). However, it should be noted that the constant on the right hand side can now change from streamline to streamline. Additionally, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Z &= \frac{\mathrm{d}}{\mathrm{d}t}\oint_C \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \oint_C \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \boldsymbol{u}, \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\right\rangle \mathrm{d}s \\
&= \oint_C \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\,\mathrm{d}\boldsymbol{x} + \oint_C \boldsymbol{u}\,\mathrm{d}\boldsymbol{u} \\
&= \oint_C \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\,\mathrm{d}\boldsymbol{x} = \oint_C \left\{\langle \boldsymbol{u}, \nabla\rangle\,\boldsymbol{u}\right\}\ \mathrm{d}\boldsymbol{x} \\
&= -\oint_C \left\{\frac{1}{\varrho}\nabla p + \nabla Q\right\}\,\mathrm{d}\boldsymbol{x} \\
&= -\oint_C \nabla\left(\Theta + Q\right)\ \mathrm{d}\boldsymbol{x} = 0\ .
\end{aligned}$$

This is *Kelvin's theorem*,[19)] which says that the circulation along a closed curve in an inviscid barotropic flow does not change over time.

If an irrotational flow with an arbitrary type of incompressibility is considered, (1.28) leads together with (1.25) to

$$\Delta\phi = 0 \tag{1.32}$$

whereas, for a compressible fluid in the case of stationary flow, the continuity equation (1.4) is

$$\operatorname{div}(\varrho\boldsymbol{u}) = 0\ .$$

Let us additionally assume the flow to be barotropic. Then, because of

$$\operatorname{div}(\varrho\boldsymbol{u}) = \varrho\operatorname{div}\boldsymbol{u} + \frac{\mathrm{d}\varrho}{\mathrm{d}p}\,\langle\nabla p, \boldsymbol{u}\rangle$$

19) Lord Kelvin of Largs (1824–1907); Glasgow and because of

$$\hat{c} = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\varrho}} \quad \text{(cf. (1.67))} ,$$

(1.26) leads to

$$\left(1 - \left(\frac{u_1}{\hat{c}}\right)^2\right) \partial_{xx}\phi + \left(1 - \left(\frac{u_2}{\hat{c}}\right)^2\right) \partial_{yy}\phi + \left(1 - \left(\frac{u_3}{\hat{c}}\right)^2\right) \partial_{zz}\phi$$
$$- 2\left(\frac{u_1\, u_2}{\hat{c}^2} \cdot \partial_y u_1 + \frac{u_2\, u_3}{\hat{c}^2} \cdot \partial_z u_2 + \frac{u_3\, u_1}{\hat{c}^2} \cdot \partial_x u_3\right) = 0 , \tag{1.33}$$

as far as the exterior forces vanish. In this situation, (1.33) generalizes (1.32).

Obviously, (1.33) is a quasilinear partial differential equation of second order for ϕ, which is certainly elliptic if

$$M := \frac{\|\boldsymbol{u}\|}{\hat{c}} < 1 , \tag{1.34}$$

i.e., in areas of *subsonic flow*.

Definition

M is called the *Mach number*.[20]

In particular, if we consider a constant flow in the x-direction which is only disturbed in the neighborhood of a slim airfoil[21] with a small angle of attack, the products of the values u_i $(i = 2, 3)$ with each other and with u_1 can be neglected when compared with 1. This leads to $\|\boldsymbol{u}\|^2 = u_1^2$ and to a shortened version of (1.33), namely to

$$\left(1 - \left(\frac{u_1}{\hat{c}}\right)^2\right) \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0 . \tag{1.35}$$

This equation is hyperbolic in areas of *supersonic flow* ($M > 1$), and the full equation (1.33) is also hyperbolic in this case.

Let us extend our idealizations by assuming that the flow under consideration is a two-dimensional *plane flow*. This means that one of the components of the velocity vector $\boldsymbol{u}$ in a rectangular coordinate system, e.g., the component in the z-direction, vanishes for all $(x, y) \in \mathbb{R}^2$ and for all $t \geq 0$:

$$u_3 = 0 . \tag{1.36}$$

In the case of a two-dimensional supersonic flow along a slim airfoil for which (1.35) holds, *Mach's angle* β, determined from

$$|\sin\beta| = \frac{1}{M} < 1 ,$$

20) Ernst Mach (1838–1916); Graz, Praha, Vienna

21) A cross-section of a wing or another rigid body in a plane parallel to the direction of the flow

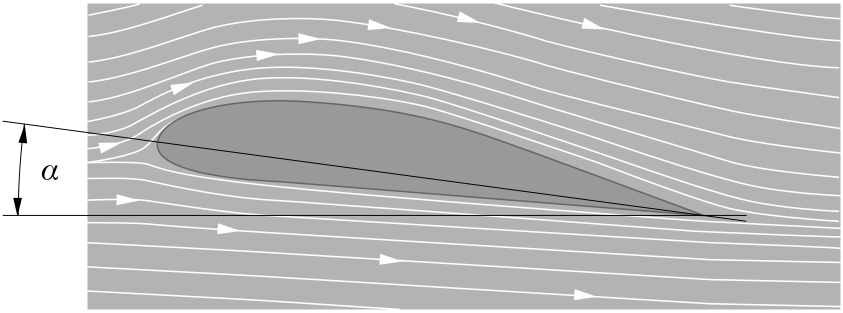

Fig. 1.3 Flow around a wing if the angle of attack is small.

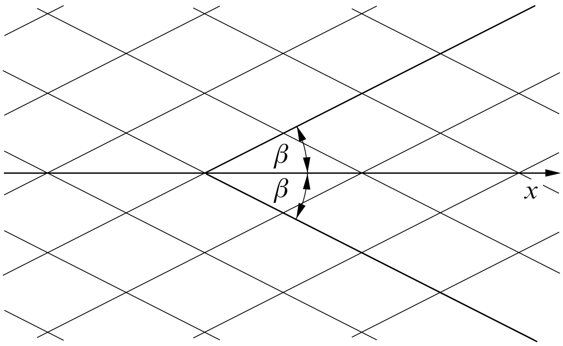

Fig. 1.4 Mach's net in the case of a linearized supersonic flow ($M = \text{const} > 1$).

describes the angle between the characteristics of the wave equation

$$(1 - M^2)\partial_{xx}\phi + \partial_{yy}\phi = 0$$

and the flow direction given by the direction of the x-axis. The set of all of these characteristics is called *Mach's net*. This net plays an important role when so-called *methods of characteristics* are used in order to establish efficient numerical procedures.

The two-dimensional case of rotational symmetry, e.g., flow along a projectile, leads correspondingly to *Mach's cone*.

In the two-dimensional plane situation, (1.29) becomes

$$\operatorname{curl} \boldsymbol{u} = \begin{pmatrix} -\partial_z u_2 \\ \partial_z u_1 \\ \partial_x u_2 - \partial_y u_1 \end{pmatrix} = 0 \, .$$

Thus, u_1 and u_2 are independent of the third spatial variable z:

$$\boldsymbol{u} = \begin{pmatrix} u_1(x, y) \\ u_2(x, y) \\ 0 \end{pmatrix}$$

with

$$\partial_x u_2 - \partial_y u_1 = 0 \; . \qquad (1.37)$$

After introducing the vector

$$\boldsymbol{v} := \begin{pmatrix} -u_2(x,y) \\ u_1(x,y) \\ 0 \end{pmatrix} \qquad (1.38)$$

and using the continuity equation (1.25), we obtain

$$\operatorname{curl} \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ \partial_x u_1 + \partial_y u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{div} u \end{pmatrix} = 0 \; ,$$

such that $\boldsymbol{v}$ can also be derived from the potential in simply connected parts of the fluid area. In other words, there is a scalar function $\psi = \psi(x, y)$ with

$$-u_2 = \partial_x \psi \; , \quad u_1 = \partial_y \psi \; . \qquad (1.39)$$

ψ is called the *stream function*. (1.28) leads to

$$\partial_x \phi = \partial_y \psi \; , \quad \partial_y \phi = -\partial_x \psi \; . \qquad (1.40)$$

Obviously, (1.40) can be interpreted as the system of *Cauchy–Riemann equations* of the complex function

$$\Omega(z) := \phi(x, y) + i\,\psi(x, y) \qquad (1.41)$$

depending on the complex variable $z = x + i\,y$.

Ω is called the *complex velocity potential* of the plane potential flow under consideration.

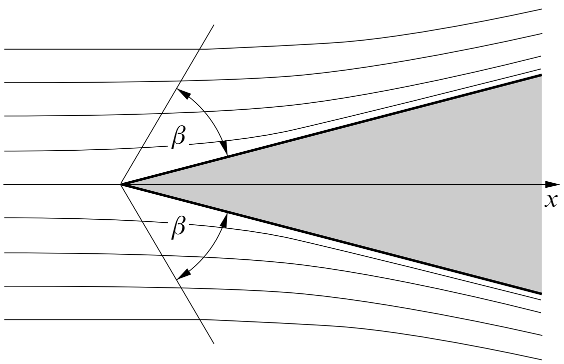

Fig. 1.5 Linearized supersonic flow around a slim rotatory cone.

We are going to assume that the first partial derivatives of the functions ϕ and ψ are continuous such that Ω is found to be a holomorphic function, and we intend to study the forces acting on rigid bodies when dipped into such a fluid flow.

Because we reduced reality to a plane flow, the rigid body is assumed to be very long with respect to the direction of the third spatial variable; more precisely, it must be of infinite length from the point of view of mathematics. Therefore, the flow around the contour of the cross-section of the body within the (x, y)-plane, e.g., around the contour of an airfoil of a long wing, is of interest.

We stated that the circulation $Z = \oint_C \boldsymbol{u}\, d\boldsymbol{x}$ vanishes as long as the closed contour C is the boundary of a simply connected domain within the fluid area in the case of a potential flow. Because of our plane model, C represents a simple closed contour in the (x, y)-plane, i.e., in $\mathbb{R}^2$.

Now, consider the situation where the boundary Γ of a rigid body is dipped into the fluid, e.g., a wing. If its airfoil is part of the area surrounded by C, this interior domain of C is no longer a simply connected domain of the fluid area. Hence, the circulation Z around the airfoil does not necessarily vanish but is found to fulfill

$$\oint_C \boldsymbol{u}\, d\boldsymbol{x} = \oint_\Gamma \boldsymbol{u}\, d\boldsymbol{x}\,.$$

In order to proof this relation, we take into account that Γ and C can, in a first step, be connected by two auxiliary lines in such a way that two simply connected domains G_1 and G_2, with contours C_1 and C_2, respectively, will occur. Because of curl $\boldsymbol{u} = 0$ in G_1 as well as in G_2, we see in a second step that

$$\oint_{C_1} \boldsymbol{u}\, d\boldsymbol{x} = 0 \quad \text{as well as} \quad \oint_{C_2} \boldsymbol{u}\, d\boldsymbol{x} = 0$$

holds. This leads to

$$\oint_{C_1} \boldsymbol{u}\, d\boldsymbol{x} + \oint_{C_2} \boldsymbol{u}\, d\boldsymbol{x} = 0$$

(cf. Fig. 1.6).

We realize in a third step that the integrations back and forth along each of the auxiliary lines extinguish each other such that

$$\oint_{C_1} \boldsymbol{u}\, d\boldsymbol{x} + \oint_{C_2} \boldsymbol{u}\, d\boldsymbol{x} = \oint_C \boldsymbol{u}\, d\boldsymbol{x} + \oint_{-\Gamma} \boldsymbol{u}\, d\boldsymbol{x}$$

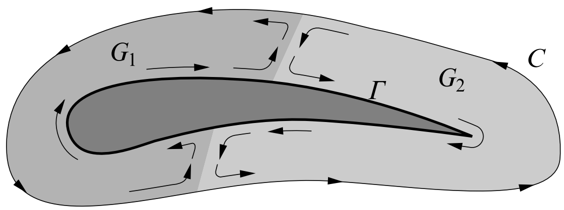

Fig. 1.6 Auxiliary step in the computation of the buoyancy generated by plane potential flows.

results. Here, $-\Gamma$ denotes the contour of the rigid body along the negative direction, and this ends the proof.

Remark

It should be noted that the proof given here is simplified because Γ not only passes through the interior of the domain occupied by the fluid but is also part of its boundary. As a matter of fact, one must first, roughly speaking, investigate the case where Γ is replaced by a line Γ_ε of distance ε from Γ that passes only through the fluid, and in a next step one must study the limit situation $\varepsilon \to 0$.

Let Γ now be parametrized by $\boldsymbol{r} = \boldsymbol{r}(\tau)\,,\ 0 \le \tau \le T$. Then

$$Z = \int_0^T \langle \boldsymbol{u}(\boldsymbol{r}(\tau)), \dot{\boldsymbol{r}}(\tau)\rangle \; \mathrm{d}\tau = \int_0^T \{u_1\,\dot{x} + u_2\,\dot{y}\} \; \mathrm{d}\tau \tag{1.42}$$

holds, and $\dot{\boldsymbol{r}} = \begin{pmatrix}\dot{x}(\tau)\\ \dot{y}(\tau)\end{pmatrix}$ is a vector tangential to the curve Γ at the point $(x(\tau), y(\tau))$ such that

$$\boldsymbol{n} := \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} \tag{1.43}$$

is a normal unit vector of Γ at this point.

According to (1.13), we assume that the velocity of the flow at the surface of the rigid body is tangential to this surface:

$$\langle \boldsymbol{u}, \boldsymbol{n}\rangle = 0 \quad \text{along} \quad \Gamma\,,$$

i.e., $u_1\,\dot{y} = u_2\,\dot{x}$. From this, and with (1.42), the circulation becomes

$$Z = \int_0^T (u_1 - i\,u_2)\,(\dot{x} + i\,\dot{y})\; \mathrm{d}\tau\,,$$

i.e.,

$$Z = \int_\Gamma w(z)\; \mathrm{d}z\,, \tag{1.44}$$

with

$$w(z) := u_1(x,y) - i\,u_2(x,y)\,. \tag{1.45}$$

$w(z)$ is a holomorphic function in the whole domain outside the airfoil because (1.39) yields

$$\partial_x u_1 = \partial_{xy}\psi = \partial_{yx}\psi = \partial_y(-u_2)\,,$$

and because (1.28) leads to

$$\partial_y u_1 = \partial_{xy}\phi = \partial_{yx}\phi = -\partial_x(-u_2)\,,$$

the Cauchy–Riemann equations for the function $w(z)$ are fulfilled.

Hence, also with respect to the computation of the circulation, integration along C instead of Γ is permitted:

$$Z = \int_C w(z)\, \mathrm{d}z\ . \tag{1.46}$$

Here, we choose C in such a way that it lies in the annulus between two concentric circles, where circle K_r of radius r surrounds the airfoil and K_R is a circle with a sufficiently great arbitrary radius $R > r$. Without any loss of generality, we assume the center of each circle to be the origin of the former (x, y)-plane, which now becomes the complex z-plane.

$w(z)$ can be represented in the annulus (and therefore for every $z \in C$ in particular) by a Laurent series[22] around the center $z_0 = 0$ (cf. Fig. 1.7):

$$w(z) = \sum_{\nu=-\infty}^{+\infty} a_\nu\, z^\nu\ .$$

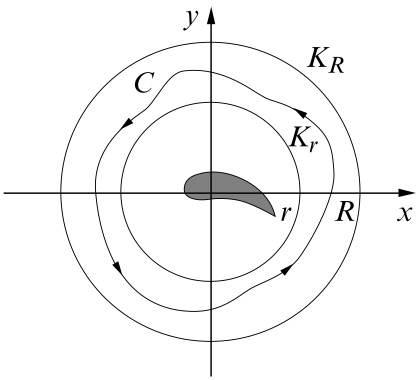

Fig. 1.7 Annulus around an airfoil.

We know from experience that the fluid flow is often only influenced by the rigid body within the neighborhood (which has a certain size) of the contour; the flow around a ship that crosses a calm lake at a constant velocity provides an example of this. From this point of view, we are going to assume that the velocity $\boldsymbol{u} = \boldsymbol{u}(x, y)$ is constant for $|z| \to \infty$:

$$\lim_{|z|\to\infty} \boldsymbol{u}(x, y) = \begin{pmatrix} u_{1,\infty} \\ u_{2,\infty} \end{pmatrix} \tag{1.47}$$

with constant components $u_{1,\infty}$ and $u_{2,\infty}$ such that

$$\lim_{|z|\to\infty} w(z) = u_{1,\infty} - i\, u_{2,\infty}\ .$$

22) Pierre Alphonse Laurent (1813–1854); Le Havre

Because the coefficients a_ν ($\nu = 0, \pm 1, \pm 2, \ldots$) are constant numbers which do not depend on R, and because R is allowed to tend to infinity, we find for $|z| \to \infty$ that all the a_ν with positive indices must vanish.

Hence, the result

$$\lim_{|z|\to\infty} w(z) = a_0 = u_{1,\infty} - i\, u_{2,\infty} \tag{1.48}$$

follows.

On the other hand, Cauchy's formula

$$a_{-1} = \frac{1}{2\pi i} \oint_C w(\zeta)\, \mathrm{d}\zeta$$

leads to

$$Z = 2\pi i\, a_{-1} \ . \tag{1.49}$$

The (two-dimensional) force $\boldsymbol{K}$ to be calculated, which is caused by the flow and acts on the rigid body, is found to be:

$$\boldsymbol{K} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = -\int_0^L p\, \boldsymbol{n}\, \mathrm{d}s \quad \left([\boldsymbol{K}] = \frac{[\text{force}]}{[\text{length}]} \right) , \tag{1.50}$$

where $\boldsymbol{n}$ is the unit normal vector from (1.43), p denotes the pressure along the surface of the body caused by the fluid, s measures the arc length along Γ beginning at an arbitrary point on it, and where L is the total length of Γ. Let us study this force separately from the other exterior forces acting on the body; i.e., let us assume that the sum of these other forces vanishes. This leads to a constant potential Q in Bernoulli's equation (1.31) and therefore to a constant *total pressure*

$$p_0 := \frac{\varrho}{2}\, \|\boldsymbol{u}\|^2 + p \ .$$

Because Γ is a closed contour, $\int_0^L \boldsymbol{n}\, \mathrm{d}s = 0$.[23)]

Therefore,

$$\boldsymbol{K} = -\int_0^L \left\{ p_0 - \frac{\varrho}{2}\, \|\boldsymbol{u}\|^2 \right\} \boldsymbol{n}\, \mathrm{d}s = \frac{\varrho}{2} \int_0^L \|\boldsymbol{u}\|^2\, \boldsymbol{n}\, \mathrm{d}s \ ,$$

23) $\dfrac{\mathrm{d}s}{\mathrm{d}\tau} = \sqrt{\dot{x}^2 + \dot{y}^2}$, hence

$$\int_0^L \boldsymbol{n}\, \mathrm{d}s = \int_0^T \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} \frac{\mathrm{d}s}{\mathrm{d}\tau}\, \mathrm{d}\tau = \begin{pmatrix} y(T) - y(0) \\ -x(T) + x(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.,

$$K_1 = \frac{\varrho}{2} \int_0^T \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \|\boldsymbol{u}\|^2 \, \dot{y} \, \frac{\mathrm{d}s}{\mathrm{d}\tau} \, \mathrm{d}\tau \, ,$$

$$K_2 = -\frac{\varrho}{2} \int_0^T \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \|\boldsymbol{u}\|^2 \, \dot{x} \, \frac{ds}{d\tau} \, \mathrm{d}\tau \, .$$

The temporary introduction of the complex number

$$k = K_2 + i\, K_1 \tag{1.51}$$

yields

$$k = -\frac{\varrho}{2} \int_0^T \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \|\boldsymbol{u}\|^2 \, (\dot{x} - i\,\dot{y}) \, \frac{\mathrm{d}s}{\mathrm{d}\tau} \, \mathrm{d}\tau = -\frac{\varrho}{2} \int_0^T \|\boldsymbol{u}\|^2 \, (\dot{x} - i\,\dot{y}) \, \mathrm{d}\tau \, .$$

The complex number within the parentheses on the right hand side of

$$\begin{aligned} \|\boldsymbol{u}\|^2 (\dot{x} - i\,\dot{y}) &= \left(u_1^2 + u_2^2\right) \, (\dot{x} - i\,\dot{y}) = (u_1 - i\,u_2)\,(u_1 + i\,u_2)\,(\dot{x} - i\,\dot{y}) \\ &= w(z) \, (u_1\,\dot{x} + u_2\,\dot{y} + i\,u_2\,\dot{x} - i\,u_1\,\dot{y}) \end{aligned}$$

is not really complex because the imaginary part vanishes (cf. (1.13)). It can therefore be replaced by its conjugate complex number:

$$\begin{aligned} \|\boldsymbol{u}\|^2 (\dot{x} - i\,\dot{y}) &= w(z) \, (u_1\,\dot{x} + u_2\,\dot{y} - i\,u_2\,\dot{x} + i\,u_1\,\dot{y}) \\ &= w(z) \, (u_1 - i\,u_2) \, (\dot{x} + i\,\dot{y}) = w^2(z)\,\dot{z} \, . \end{aligned}$$

This leads to

$$k = -\frac{\varrho}{2} \int_0^T w^2(z)\,\dot{z}\,\mathrm{d}\tau = -\frac{\varrho}{2} \int_\Gamma w^2(z)\,\mathrm{d}z \, .$$

Because w is holomorphic, w^2 is also a holomorphic function. Hence,

$$k = -\frac{\varrho}{2} \int_C w^2(z)\,\mathrm{d}z \, . \tag{1.52}$$

However, within the annulus we have

$$\begin{aligned} w^2(z) &= \left(a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots\right) \left(a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots\right) \\ &= a_0^2 + 2\,\frac{a_0\,a_{-1}}{z} + \frac{A_{-2}}{z^2} + \frac{A_{-3}}{z^3} + \ldots \end{aligned}$$

with certain coefficients A_ν ($\nu = -2, -3, \ldots$).

Cauchy's residuum formula and (1.52) therefore lead to

$$k = -\frac{\varrho}{2}\, 2a_0\, a_{-1} \cdot 2\pi\, i\,.$$

Thus, (1.48) and (1.49) yield

$$k = -\varrho\, \left(u_{1,\infty} - i\, u_{2,\infty}\right)\, Z\,,$$

and because the circulation Z is real (cf. (1.27)), comparison of the real and imaginary parts of (1.51) results in

$$\begin{aligned} K_1 &= \varrho\, u_{2,\infty}\, Z \\ K_2 &= -\varrho\, u_{1,\infty}\, Z\,. \end{aligned} \tag{1.53}$$

In particular, if $u_{2,\infty} = 0$ but $u_{1,\infty} \neq 0$, i.e., if the undisturbed flow is parallel to the x-axis, and if $Z \neq 0$, we obtain a *lift* $K_2 \neq 0$, i.e., there is a force acting on the rigid body perpendicular to the direction of the flow.

By using an appropriate wing construction, the airfoils can yield $Z < 0$, so that an aircraft can lift its own weight, a hydrofoil can rise out of the water, etc. Of course, Archimedes' static buoyancy, as given by (1.4), must also be taken into account.

Remark

The formulae (1.53) are called *Kutta–Zhukovsky buoyancy formulae.*[24)]

Remark

Whereas the first equation in (1.53) describes the buoyancy at least qualitatively in a correct manner, the result $K_1 = 0$ contradicts reality. Also, in the case of an incompressible irrotational stationary flow (as more or less realized by calmly flowing streams), the flow will apply some force to the rigid body (e.g., a bridge pier) parallel to the direction of flow. Of course, this contradiction results from one of our idealizations: the assumption of an inviscid and therefore frictionless fluid.

In order to understand what really happens parallel to the flow, we must reduce the amount of idealization. This will later be achieved by using the so-called Navier–Stokes equations rather than the Euler equations, and the so-called *no-slip condition* given below rather than (1.13).

$$\boldsymbol{u} = 0 \quad \text{along} \quad \Gamma\,. \tag{1.54}$$

Equation (1.54) expresses the idea that moving viscous fluids leave a monomolecular stationary layer on impermeable walls of solid bodies because of adhesion. Thus, friction along such surfaces does not mean friction between the fluid and the solid material, but rather friction between fluid particles.

Of course, there are also other boundary conditions where *partial slip* occurs on the surface, e.g., in rarefied gas flow, with porous walls, etc. The tangential component of the flow is then proportional to the local shear stress.

24) Martin Wilhelm Kutta (1867–1927); Stuttgart;
Nikolai Jegorowitsch Zhukovsky (1847–1921);
Moscow

If (1.33) is reduced to the case of an irrotational, stationary, plane flow (i.e., $u_3 = 0$, $\partial_y u_1 = \partial_x u_2$), if there are no exterior forces, and if (u_1, u_2) is for convenience replaced with (u, v), we get:

$$\left(1 - \left(\frac{u}{\hat{c}}\right)^2\right) \partial_x u + \left(1 - \left(\frac{v}{\hat{c}}\right)^2\right) \partial_y v - \frac{u\,v}{\hat{c}^2} \cdot \left(\partial_y u + \partial_x v\right) = 0 \,. \tag{1.55}$$

The uv-plane is called the *hodograph plane*.

We assume the equations

$$u = u(x, y)$$
$$v = v(x, y)$$

to be invertible such that x and y can be expressed by u and v, i.e.,

$$D := \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} \neq 0 \,.$$

In other words, it will be assumed that the vectors $\partial_x \boldsymbol{u}$ and $\partial_y \boldsymbol{u}$ are linearly independent.

This leads immediately to

$$\partial_x u = D\,\partial_v y \,, \quad \partial_y u = -D\,\partial_v x \,, \quad \partial_x v = -D\,\partial_u y \,, \quad \partial_y v = D\,\partial_u x \,.$$

The nonlinear equation (1.55) together with the equation $\partial_y u = \partial_x v$ for the functions $u(x, y)$ and $v(x, y)$ of the irrotational flow can therefore – after division by D – be transformed into the linear equations

$$\left(1 - \frac{v^2}{\hat{c}^2}\right) \partial_u x + \left(1 - \frac{u^2}{\hat{c}^2}\right) \partial_v y + \frac{uv}{\hat{c}^2} \; (\partial_v x + \partial_u y) = 0$$
$$\partial_v x - \partial_u y = 0 \tag{1.56}$$

for the functions $x(u, v)$ and $y(u, v)$.

The transition from the original equations to the linear equations (1.56) is called the *method of hodographs* and corresponds to the *Legendre transformation* in the theory of partial differential equations.

In simply connected domains, $\boldsymbol{x}(\boldsymbol{u})$ can be derived from a potential Θ (i.e., $x = \Theta_u$, $y = \Theta_v$), because of

$$\operatorname{curl} \boldsymbol{x}(\boldsymbol{u}) = (0, 0, \partial_u y - \partial_v x)^T = 0 \,.$$

Equation (1.56) therefore yields

$$\left(1 - \frac{v^2}{\hat{c}^2}\right) \partial_{uu}\Theta + \left(1 - \frac{u^2}{\hat{c}^2}\right) \partial_{vv}\Theta + 2\frac{uv}{\hat{c}^2} \partial_{uv}\Theta = 0 \,. \tag{1.57}$$

If polar coordinates ($w = \|\boldsymbol{u}\|$, α) are used in the hodograph plane, i.e.,

$$u = w \cos\alpha \,, \quad v = w \sin\alpha \,,$$
$$\partial_u = \cos\alpha\,\partial_w - \frac{1}{w} \sin\alpha\,\partial_\alpha \,,$$
$$\partial_v = \sin\alpha\,\partial_w + \frac{1}{w} \cos\alpha\,\partial_\alpha \,,$$

Equation (1.57) becomes the so-called *hodograph equation*:

$$\partial_{ww}\Theta + \frac{1}{w^2}\left(1-\frac{w^2}{\hat{c}^2}\right)\partial_{\alpha\alpha}\Theta + \frac{1}{w}\left(1-\frac{w^2}{\hat{c}^2}\right)\partial_w\Theta = 0 \tag{1.58}$$

which does not explicitly contain α.

If we try to solve (1.58) by the ansatz

$$\Theta(w,\alpha) = g(w)\sin(m\alpha) \quad \text{or} \quad \Theta(w,\alpha) = g(w)\cos(m\alpha)\,, \quad (m \in \mathbb{R})\,, \tag{1.59}$$

we obtain for the unknown function $g(w)$ the ordinary differential equation

$$g''(w) + \frac{1}{w}\left(1-\frac{w^2}{\hat{c}^2}\right)g'(w) - \frac{m^2}{w^2}\left(1-\frac{w^2}{\hat{c}^2}\right)g(w) = 0\,. \tag{1.60}$$

Solutions of the hodograph equation of type (1.59) are called *Chapligin solutions.*[25)]

Let $g_m(w)$ be a solution of (1.60) that belongs to a particular m and let $\Theta_m(w,\alpha)$ be the solution of (1.59) that corresponds to this solution.

Examples:

$m = 0$ leads to

$$\frac{g_0''}{g_0'} = -\frac{1}{w} + \frac{w}{\hat{c}^2}\,,$$

hence

$$g_0' = \frac{c_1^{(0)}}{w}e^{\frac{1}{2}\frac{w^2}{\hat{c}^2}}\,.$$

The power series of $e^{\frac{1}{2}\frac{w^2}{\hat{c}^2}}$ converges uniformly for all values of w. Integration can therefore be performed term-by-term, yielding

$$g_0(w) = c_1^{(0)}\left\{\ln w + \sum_{\nu=1}^{\infty}\frac{\left(\frac{w}{\hat{c}}\right)^{2\nu}}{2^{\nu+1}\nu\nu!}\right\} + c_2^{(0)}\,.$$

Here, $c_1^{(0)}$ and $c_2^{(0)}$ are arbitrary constants.

Analogously for $m = 1$:

$$g_1(w) = c_1^{(1)}w + c_2^{(1)}\left\{\frac{1}{w^2} + \frac{w\ln w}{2\hat{c}^2} + \sum_{\nu=0}^{\infty}\frac{w^{2(\nu+1)}}{2^{\nu+2}\hat{c}^{2(\nu+2)}}\right\}\,.$$

Because (1.58) is linear and homogeneous, all linear combinations of the particular Chapligin solutions also solve Eq. (1.58). The coefficients of the expansion as well as the constants $c_1^{(m)}$ and $c_2^{(m)}$ must be chosen in such a way that the expansion fits the given situation, at least approximately.

25) C.A. Chapligin: Sci. Ann. Univ. Moscow. Math. Phys. **21** (1904) 1–121

1.4 Motionless Fluids and Sound Propagation

Obviously, because viscosity only plays a role in moving fluids, the results of this section are also valid for real fluids.

In a first step, let us consider the case for constant density ϱ, which is approximately realized in liquids.

In this situation, (1.26) leads for motionless fluids (i.e., for $\boldsymbol{u} = 0$) to the so-called *hydrostatic equation*

$$\hat{\boldsymbol{k}} = \nabla \left(\frac{p}{\varrho} \right) ,$$

and (1.31) becomes

$$Q + \frac{p}{\varrho} = \text{const} .$$

Let us assume that we do not yet know how the free surface of a motionless liquid behaves if only the force of gravity and a constant exterior (e.g., atmospheric) pressure p_0 affects this liquid. Hence, the force per mass unit is given by

$$\hat{\boldsymbol{k}} = (0,\ 0,\ -g)^T \quad (g = \text{acceleration due to gravity}) ,$$

such that $Q = gz + \text{const}$ (i.e., $\varrho g z + p = \text{const}$). In particular, if (x, y, z_0) is a point on the surface, $\varrho g z + p = \varrho g z_0 + p_0$ or

$$\varrho g (z - z_0) = -(p - p_0) = -\hat{p} \tag{1.61}$$

holds. Here, $\hat{p}$ is the overpressure inside the liquid compared with the exterior pressure.

Because of

$$z_0 = \frac{\text{const} - p_0}{\varrho g} ,$$

z_0 is constant, and thus independent of (x, y). In other words, the surface of the liquid is a plane, or more precisely it is parallel to the Earth's surface. If $h = z_0 - z$ is the height of an arbitrarily shaped liquid column, (1.61) gives

$$\varrho g h q = \hat{p} q ,$$

where q is the base area of the column.

The force affecting this base is given by the right hand side of the equation and does not depend on the form of the column, whereas the left hand side gives the weight of a *cylindrical* column of the fluid with the same base area and the same height.

This phenomenon is called the *hydrostatic paradox*.

We are now going to dip a solid body of volume V and surface F into a stationary liquid.

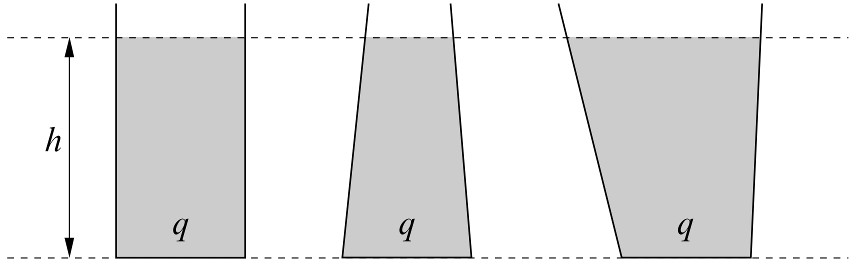

Fig. 1.8 Hydrostatic paradox.

Obviously, an overall force $\boldsymbol{K}$ affects the body, where

$$\boldsymbol{K} = (K_1, K_2, K_3)^T = -\int\limits_F p\,\boldsymbol{n}\,\mathrm{d}o + \boldsymbol{G}\,.$$

Here, p is the interior pressure of the liquid, $\boldsymbol{G}$ is the weight of the body, and $\boldsymbol{n}$ denotes the outward-directed normal unit vector at the points on the surface. Because of $\boldsymbol{G} = (0\,,\ 0\,,\ -G)^T$, (1.61) yields

$$K_1 = \varrho\, g \int\limits_F \left(z - z_0 - \frac{p_0}{\varrho\, g}\right) n_1\,\mathrm{d}o = \varrho\, g \int\limits_F \langle \boldsymbol{a}, \boldsymbol{n}\rangle\ \mathrm{d}o$$

with $\boldsymbol{a} := (z - z_0 - \dfrac{p_0}{\varrho\, g}, 0, 0)^T$. The divergence theorem therefore leads to

$$K_1 = \varrho\, g \int\limits_V \operatorname{div} \boldsymbol{a}\,\mathrm{d}V = \varrho\, g \int\limits_V \frac{\partial(z - z_0)}{\partial x}\,\mathrm{d}V = 0\,.$$

Analogously, $K_2 = 0$. K_3 is found to be given by

$$K_3 = \varrho\, g \int\limits_V \frac{\partial(z - z_0)}{\partial z}\,\mathrm{d}V - G = \varrho\, g\, V - G\,,$$

so that

$$\boldsymbol{K} = (0, 0, \varrho\, g\, V - G)^T\,. \qquad (1.62)$$

ϱ is the density of the liquid (!), so $\varrho\, g\, V$ is the weight of the particular part of the liquid which is displaced by the solid body. Hence, the body is affected by a force directed against the direction of action of its weight, and this (static) buoyancy equals the weight of the displaced quantity of the liquid (*Archimedes' principle*).[26]

In order to study the propagation of sound in a fluid, we assume that the fluid does not move from a macroscopic point of view. Moreover, we will only consider the sound effects resulting from very small changes in the density and pressure

26) Archimedes (about 220 BC); Syracuse within the fluid. Therefore, we no longer assume the density to be constant, but we assume only small variations in it.

Let $\hat{\varrho}$ and $\hat{p}$ be the averages of the density and the pressure, respectively. Only these averages are expected to be constant quantities.

The density disturbances will be expressed by

$$\varrho = \hat{\varrho}\ \big(1 + \sigma(\boldsymbol{x}, t)\big)$$

where $|\sigma| \ll 1$ and the spatial derivatives of σ are small.[27]

The continuity equation (1.4) then becomes

$$\hat{\varrho}\,\sigma_t + \hat{\varrho}\ \operatorname{div}\big((1+\sigma)\boldsymbol{u}\big) = 0\ ,$$

and so, after dividing by $\hat{\varrho}$ and taking the assumptions for σ into account,

$$\sigma_t + \operatorname{div}\boldsymbol{u} = 0\ . \tag{1.63}$$

Experiments show that the propagation of sound waves occurs more or less adiabatically, i.e., without any gain or loss of heat. The equation of state to be taken into account is therefore

$$\varrho^{-\gamma}\,p = \text{const}^{28)} \tag{1.64}$$

such that

$$\frac{p}{\hat{p}} = \left(\frac{\varrho}{\hat{\varrho}}\right)^{\gamma} = (1+\sigma)^{\gamma}\ .$$

Because σ is small,

$$(1+\sigma)^{\gamma} \approx 1 + \gamma\,\sigma\ .$$

Hence

$$p = \hat{p}(1 + \gamma\,\sigma)\ ,$$

so that

$$\frac{1}{\varrho}\,\nabla p = \frac{\hat{p}\,\gamma}{(1+\sigma)\,\hat{\varrho}}\,\nabla\sigma\ . \tag{1.65}$$

In our model of small disturbances, the velocity of the fluid particles and its spatial derivatives are so small that higher-order terms of these quantities can be neglected

27) Our results do not necessarily hold in the case of the propagation of *large* variations in the pressure or density, such as can occur in the case of detonations.

28) γ from (1.9)

compared with first-order terms. In other words, the convection term in (1.6) can be neglected. Also, exterior forces do not play a role in our scenario.

If (1.65) is then put into (1.6), we find

$$\boldsymbol{u}_t = -\frac{\gamma\,\hat{p}}{\hat{\varrho}}\,\nabla\sigma\ ,$$

where $1 + \sigma$ was approximated by 1.

Forming the divergence of this term, and then changing the sequence of the time derivative and the spatial derivatives, we end up with

$$(\operatorname{div}\boldsymbol{u})_t = -\frac{\gamma\,\hat{p}}{\hat{\varrho}}\,\Delta\sigma\ .$$

This result can be compared with (1.63) when differentiated with respect to t. This comparisons yields

$$\sigma_{tt} = \frac{\gamma\,\hat{p}}{\hat{\varrho}}\,\Delta\,\sigma\ ,$$

a wave equation for σ. It shows that the sound waves propagate within the fluid with the velocity

$$\hat{c} = \sqrt{\frac{\gamma\,\hat{p}}{\hat{\varrho}}}\ . \tag{1.66}$$

Definition

$\hat{c}$ is called the *local speed of sound.*

Remark

Because of (1.64), $\hat{c}$ can also be represented by

$$\hat{c} = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\varrho}}\ . \tag{1.67}$$