# 1 Analytic Functions

**Abstract.** We introduce the theory of functions of a complex variable. Many familiar functions of real variables become multivalued when extended to complex variables, requiring branch cuts to establish single-valued definitions. The requirements for differentiability are developed and the properties of analytic functions are explored in some detail. The Cauchy integral formula facilitates development of power series and provides powerful new methods of integration.

## 1.1 Complex Numbers

### 1.1.1 Motivation and Definitions

The definition of complex numbers can be motivated by the need to find solutions to polynomial equations. The simplest example of a polynomial equation without solutions among the real numbers is $z^2 == -1$. Gauss demonstrated that by defining two solutions according to

$$z^2 == -1 \Longrightarrow z = \pm i \tag{1.1}$$

one can prove that any polynomial equation of degree $n$ has $n$ solutions among complex numbers of the form $z = x + iy$ where $x$ and $y$ are real and where $i^2 = -1$. This powerful result is now known as the *fundamental theorem of algebra*. The object $i$ is described as an *imaginary number* because it is not a real number, just as $\sqrt{2}$ is an irrational number because it is not a rational number. A number that may have both real and imaginary components, even if either vanishes, is described as *complex* because it has two parts. Throughout this course we will discover that the rich properties of functions of complex variables provide an amazing arsenal of weapons to attack problems in mathematical physics.

The complex numbers can be represented as ordered pairs of real numbers $z = (x, y)$ that strongly resemble the Cartesian coordinates of a point in the plane. Thus, if we treat the numbers $1 = (1, 0)$ and $i = (0, 1)$ as basis vectors, the complex numbers $z = (x, y) = x \times 1 + y \times i = x + iy$ can be represented as points in the complex plane, as indicated in Fig. 1.1. A diagram of this type is often called an *Argand diagram*. It is useful to define functions Re or Im that retrieve the real part $x = \mathrm{Re}[z]$ or the imaginary part $y = \mathrm{Im}[z]$ of a complex number. Similarly, the modulus, $r$, and phase, $\theta$, can be defined as the polar coordinates

$$r = \sqrt{x^2 + y^2}, \quad \theta = \mathrm{ArcTan}\left[\frac{y}{x}\right] \tag{1.2}$$

by analogy with two-dimensional vectors.

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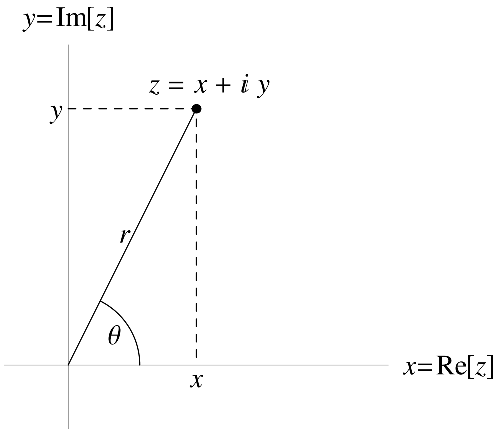


**Figure 1.1.** Cartesian and polar representations of complex numbers.

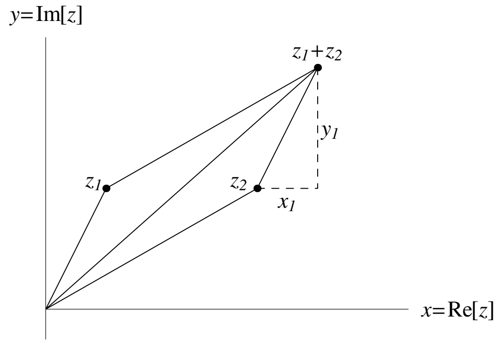


**Figure 1.2.** Addition of complex numbers.

Continuing this analogy, we also define the addition of complex numbers by adding their components, such that

$$z_1 + z_2 = \left(x_1 + x_2, y_1 + y_2\right) \Longleftrightarrow z_1 + z_2 = \left(x_1 + x_2\right) \times 1 + \left(y_1 + y_2\right) \times i \tag{1.3}$$

as diagrammed in Fig. 1.2. The complex numbers then form a *linear vector space* and addition of complex numbers can be performed graphically in exactly the same manner as for vectors in a plane.

However, the analogy with Cartesian coordinates is not complete and does not extend to multiplication. The multiplication of two complex numbers is based upon the distributive property of multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + i y_1)(x_2 + i y_2) \\ &= x_1 x_2 + i^2 y_1 y_2 + i(x_1 y_2 + x_2 y_1) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned} \tag{1.4}$$

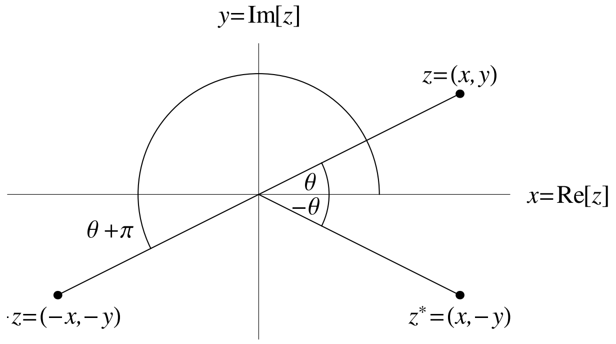


**Figure 1.3.** Inversion and complex conjugation of a complex number.

and the definition $i^2 = -1$. The product of two complex numbers is then another complex number with the components

$$z_1 z_2 = \left(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1\right) \tag{1.5}$$

More formally, the complex numbers can be represented as ordered pairs of real numbers $z = (x, y)$ with equality, addition, and multiplication defined by:

$$z_1 = z_2 \Longrightarrow x_1 = x_2 \wedge y_1 = y_2 \tag{1.6}$$

$$z_1 + z_2 = \left(x_1 + x_2, y_1 + y_2\right) \tag{1.7}$$

$$z_1 \times z_2 = \left(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1\right) \tag{1.8}$$

One can show that these definitions fulfill all the formal requirements of a *field*, and we denote the complex number field as $\mathbb{C}$. Thus, the field of real numbers is contained as a subset, $\mathbb{R} \subset \mathbb{C}$.

It will also be useful to define complex conjugation

$$\text{complex conjugation: } z = (x, y) \Longrightarrow z^* = (x, -y) \tag{1.9}$$

and absolute value functions

$$\text{absolute value: } |z| = \sqrt{x^2 + y^2} \tag{1.10}$$

with conventional notations. Geometrically, complex conjugation represents reflection across the real axis, as sketched in Fig. 1.3.

The Re, Im, and Abs functions can now be expressed as

$$\mathrm{Re}[z] = \frac{z + z^*}{2}, \quad \mathrm{Im}[z] = \frac{z - z^*}{2i}, \quad |z|^2 = zz^* \tag{1.11}$$

Thus, we quickly obtain the following arithmetic facts:

$$
\begin{aligned}
&i = (0, 1) \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \\
&\text{scalar multiplication: } c \in \mathbb{R} \Longrightarrow cz = (cx, cy) \\
&\text{additive inverse: } z = (x, y) \Longrightarrow -z = (-x, -y) \Longrightarrow z + (-z) = 0 \\
&\text{multiplicative inverse: } z^{-1} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{z^*}{|z|^2}
\end{aligned}
\tag{1.12}
$$

### 1.1.2 Triangle Inequalities

Distances between points in the complex plane are calculated using a metric function. A *metric* $d[a, b]$ is a real-valued function such that

1. $d[a, b] > 0$ for all $a \neq b$
2. $d[a, b] = 0$ for all $a = b$
3. $d[a, b] = d[b, a]$
4. $d[a, b] \leq d[a, c] + d[c, b]$ for any $c$.

Thus, the Euclidean metric $d[z_1, z_2] = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is suitable for $\mathbb{C}$. Then with geometric reasoning one easily obtains the triangle inequalities:

$$
\text{triangle inequalities : } \big||z_1| - |z_2|\big| \leq |z_1 \pm z_2| \leq |z_1| + |z_2| \tag{1.13}
$$

Note that $\mathbb{C}$ cannot be ordered (it is not possible to define $<$ properly).

### 1.1.3 Polar Representation

The function $e^{i\theta}$ can be evaluated using the power series

$$
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} (-)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-)^n \frac{\theta^{2n+1}}{(2n+1)!} = \operatorname{Cos}[\theta] + i \operatorname{Sin}[\theta] \tag{1.14}
$$

giving a result known as *Euler's formula*. Thus, we can represent complex numbers in *polar form* according to

$$
z = re^{i\theta} \Longrightarrow x = r \operatorname{Cos}[\theta], \tag{1.15}
$$

$$
y = r \operatorname{Sin}[\theta] \quad \text{with} \quad r = |z| = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arg[z] \tag{1.16}
$$

where $r$ is the *modulus* or *magnitude* and $\theta$ is the *phase* or *argument* of $z$. Although addition of complex numbers is easier with the Cartesian representation, multiplication is usually easier using polar notation where the product of two complex numbers becomes

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \big(\mathrm{Cos}[\theta_1] + i\,\mathrm{Sin}[\theta_1]\big)\big(\mathrm{Cos}[\theta_2] + i\,\mathrm{Sin}[\theta_2]\big) \\ &= r_1 r_2 \big(\mathrm{Cos}[\theta_1]\,\mathrm{Cos}[\theta_2] - \mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2] + i\big(\mathrm{Sin}[\theta_1]\,\mathrm{Cos}[\theta_2] + \mathrm{Cos}[\theta_1]\,\mathrm{Sin}[\theta_2]\big)\big) \\ &= r_1 r_2 \big(\mathrm{Cos}[\theta_1 + \theta_2] + i\,\mathrm{Sin}[\theta_1 + \theta_2]\big) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned} \tag{1.17}$$

Thus, the moduli multiply while the phases add. Note that in this derivation we did not assume that $e^{z_1} e^{z_2} = e^{z_1+z_2}$, which we have not yet proven for complex arguments, relying instead upon the Euler formula and established properties for trigonometric functions of real variables.

Using the polar representation, it also becomes trivial to prove *de Moivre's theorem*

$$\left(e^{i\theta}\right)^n = e^{in\theta} \Longrightarrow \left(\mathrm{Cos}[\theta] + i\,\mathrm{Sin}[\theta]\right)^n = \mathrm{Cos}[n\theta] + i\,\mathrm{Sin}[n\theta] \quad \text{for integer } n\,. \tag{1.18}$$

However, one must be careful in performing calculations of this type. For example, one cannot simply replace $(e^{in\theta})^{1/n}$ by $e^{i\theta}$ because the equation, $z^n == w$ has n solutions $\{z_k, k = 1, n\}$ while $e^{i\theta}$ is a unique complex number. Thus, there are $n$, $n^{\mathrm{th}}$-roots of unity, obtained as follows.

$$z = re^{i\theta} \Longrightarrow z^n = r^n e^{in\theta} \tag{1.19}$$

$$z^n == 1 \Longrightarrow r = 1, \quad n\theta = 2k\pi \tag{1.20}$$

$$\therefore z = \mathrm{Exp}\left[i\frac{2\pi k}{n}\right] = \mathrm{Cos}\left[\frac{2\pi k}{n}\right] + i\,\mathrm{Sin}\left[\frac{2\pi k}{n}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1 \tag{1.21}$$

In the Argand plane, these roots are found at the vertices of a regular $n$-sided polygon inscribed within the unit circle with the principal root at $z = 1$. More generally, the roots

$$z^n = w = \rho e^{i\phi} \Longrightarrow z_k = \rho^{1/n}\,\mathrm{Exp}\left[i\frac{\phi + 2\pi k}{n}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1 \tag{1.22}$$

of $e^{i\phi}$ are found at the vertices of a rotated polygon inscribed within the unit circle, as illustrated in Fig. 1.4.

### 1.1.4 Argument Function

The graphical representation of complex numbers suggests that we should obtain the phase using

$$\theta \stackrel{?}{=} \arctan\left[\frac{y}{x}\right] \tag{1.23}$$

but this definition is unsatisfactory because the ratio $y/x$ is not sensitive to the quadrant, being positive in both first and third and negative in both second and fourth quadrants. Consequently, computer programs using $\arctan[\frac{y}{x}]$ return values limited to the range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

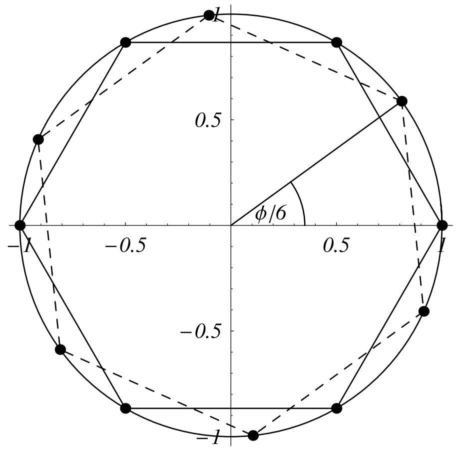


**Figure 1.4.** Solid: 6th roots of 1, dashed: 6th roots of $e^{i\phi}$.

A better definition is provided by a quadrant-sensitive extension of the usual arctangent function

$$\mathrm{ArcTan}[x, y] = \mathrm{ArcTan}\left[\frac{y}{x}\right] + \frac{\pi}{2}(1 - \mathrm{Sign}[x])\,\mathrm{Sign}[y] \tag{1.24}$$

that returns values in the range $(-\pi, \pi)$. (Unfortunately, the order of the arguments is reversed between Fortran and *Mathematica.*) Therefore, we define the *principal branch* of the argument function by

$$\mathrm{Arg}[z] = \mathrm{Arg}[x + iy] = \mathrm{ArcTan}[x, y] \tag{1.25}$$

where $-\pi < \mathrm{Arg}[z] \leq \pi$.

However, the polar representation of complex numbers is not unique because the phase $\theta$ is only defined modulo $2\pi$. Thus,

$$\arg[z] = \mathrm{Arg}[z] + 2\pi n \tag{1.26}$$

is a multivalued function where $n$ is an arbitrary integer. Note that some authors distinguish between these functions by using lower case for the multivalued and upper case for the single-valued version while others rely on context. Consider two points on opposite sides of the negative real axis, with $y \to 0^+$ infinitesimally above and $y \to 0^-$ infinitesimally below. Although these points are very close together, $\mathrm{Arg}[z]$ changes by $2\pi$ across the negative real axis.

A discontinuity of this type is usually represented by a *branch cut*. Imagine that the complex plane is a sheet of paper upon which axes are drawn. Starting at the point $(r, 0)$ one can reach any point $z = (x, y)$ by drawing a continuous circular arc of radius $r = \sqrt{x^2 + y^2}$ and we define $\arg[z]$ as the angle subtended by that arc. This function is multivalued because the circular arc can be traversed in either direction or can wind around the origin an

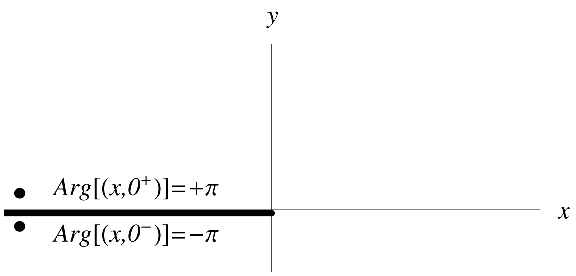


**Figure 1.5.** Branch cut for Arg[$z$].

arbitrary number of times before stopping at its destination. A single-valued version can be created by making a cut infinitesimally below the negative real axis, as sketched in Fig. 1.5, that prevents a continuous arc from subtending more than $\pm\pi$ radians. Points on the negative real axis are reached by positive (counterclockwise) arcs with $\text{Arg}[(-|z|, 0)] = \pi$ while points infinitesimally below the negative real axis can only be reached by negative arcs with $\text{Arg}[(-|z|, 0^-)] \to -\pi$. Thus, Arg[$z$] is single-valued and is continuous on any path that does not cross its branch cut, but is discontinuous across the cut.

The principal branch of the argument function is defined by the restriction $-\pi < \text{Arg}[z] \leq \pi$. Notice that one side of this range is open, represented by $<$, while the other side is closed, represented by $\leq$. This notation indicates that the cut is infinitesimally below the negative real axis, such that the argument for negative real numbers is $\pi$, not $-\pi$. This choice is not unique, but is the nearly universal convention for the argument and many related functions. The distinction between $<$ and $\leq$ many seem to be nitpicking, but attention to such details is often important in performing accurate derivations and calculations with functions of complex variables.

Many functions require one or more branch cuts to establish single-valued definitions; in fact, handling either the multivaluedness of functions of complex variables or the discontinuities associated with their single-valued manifestations is often the most difficult problem encountered in complex analysis. Although our choice of branch cut for Arg[$z$] is not unique (any radial cut from the origin to $\infty$ would serve the same purpose), it is consistent with the customary definitions of ArcTan, Log, and other elementary functions to be discussed in more detail later. The single-valued version of a function that is most common is described as its *principal branch*. For many functions there is considerable flexibility in the choice of branch cut and we are free to make the most convenient choice, provided that we maintain that choice throughout the problem. For example, in some applications it might prove convenient to define an argument function with the range $-\frac{3\pi}{4} < \text{MyArg}[z] \leq \frac{5\pi}{4}$ using the branch cut shown in Fig. 1.6. Consider the point $z_1 = (-1, -1)$ for which the standard argument function gives $\text{Arg}[z_1] = -3\pi/4$ while our new argument function gives $\text{MyArg}[z_1] = 5\pi/4$. These functions are obviously different because the same input gives different output, but both represent precisely the same ray in the complex plane. Therefore, we should consider the specification of the branch cuts as an important part of the definition of a single-valued function and recognize that different

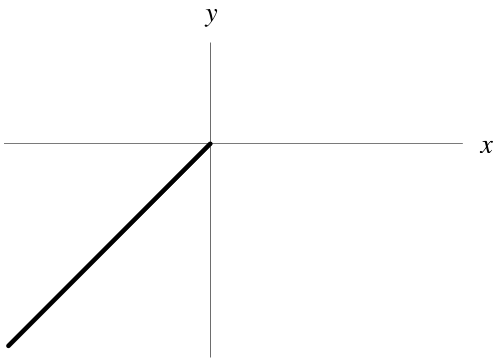


**Figure 1.6.** Branch cut for MyArg.

choices of cuts lead to related but different functions.

It is important to recognize that, because of discontinuities across branch cuts, simple algebraic relationships that apply to multivalent functions of complex variables, often do not pertain to their monovalent cousins. For example, using the polar representation of the product of two complex numbers we find

$$z_1 z_2 = r_1 r_2 \,\mathrm{Exp}\big[i(\theta_1 + \theta_2)\big] \Longrightarrow |z_1 z_2| = |z_1||z_2| \;\arg[z_1 z_2] = \arg[z_1] + \arg[z_2] \tag{1.27}$$

but this relationship for the phase does not necessarily apply to the principal branch because

$$\begin{aligned} \mathrm{Arg}[z_1 z_2] &= \arg[z_1 z_2] + 2\pi n \\ &= \arg[z_1] + \arg[z_2] - 2\pi n \\ &= \mathrm{Arg}[z_1] + \mathrm{Arg}[z_2] + 2\pi(n - n_1 - n_2) \end{aligned} \tag{1.28}$$

where $n$ must be chosen to ensure that $-\pi < \mathrm{Arg}[z_1 z_2] \leq \pi$. Often the price of single-valuedness is the awkwardness of discontinuities.

## 1.2 Take Care with Multivalued Functions

Ambiguities in the definitions of many seemingly innocuous functions require considerable care. For example, consider the common replacement

$$\sqrt{\frac{1}{z}} \overset{?}{\longleftrightarrow} z^{-1/2} \tag{1.29}$$

that one often makes without thinking. Is this apparent equivalence correct? Compare the following two methods for evaluating these quantities when $z \to -1$.

$$z = -1 \Longrightarrow z^{-1} = -1 \Longrightarrow \sqrt{\frac{1}{z}} = i \tag{1.30}$$

$$z = e^{i\pi} \Longrightarrow z^{-1/2} = e^{-i\pi/2} = -i \tag{1.31}$$

Both calculations look correct, but their results differ in sign. These expressions are not always interchangeable! One must take more care with multivalued functions.

If we represent the complex number $z$ in Cartesian form $z = x + iy$ where $x, y$ are real, then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} \tag{1.32}$$

If $x < 0$ and $y \to \varepsilon$ where $\varepsilon$ is a positive infinitesimal, then $z$ is just above and $z^{-1}$ is just below the usual cut in the square-root function (below the negative real axis). Consequently, $\mathrm{Arg}[z]$ and $\mathrm{Arg}[z^{-1}]$ differ by $2\pi$ and the arguments of $\sqrt{1/z}$ and $z^{-1/2}$ differ by $\pi$, a negative sign, in the immediate vicinity of the negative real axis. It is usually not a good idea to use the surd (square-root) symbol for complex variables – for real numbers that symbol is usually interpreted as the positive square root, but for negative or complex numbers we should employ a fractional power and define the branch cut explicitly. Then, if we define $-\pi < \mathrm{Arg}[z] \leq \pi$ with a cut infinitesimally below the negative real axis the same cut would be implied for fractional powers and the value of $z^{-1/2}$ determined using polar notation would be unambiguous on the negative real axis. Furthermore, $1/z^{1/2} = z^{-1/2}$ applies everywhere in the cut $z$-plane without the sign ambiguity encountered above. Of course, the sign discontinuity across the cut is still present – it is an essential feature of such functions.

Let us examine the square-root function, $w = f[z] = z^{1/2}$, in more detail. When $z$ is a positive real number, the square-root function maps one $z$ onto two values of $w = \pm\sqrt{x}$. Similar behavior is expected for complex $z$ because there are always two solutions to the quadratic equation $w^2 == z$. In polar notation

$$z = re^{i\theta}, \quad z == w^2 \Longrightarrow w = \sqrt{r}\,\mathrm{Exp}\left[i\frac{\theta}{2} + n\pi i\right] \tag{1.33}$$

where, by convention, $\sqrt{r}$ represents the positive square root for real numbers and where $n = 0, 1$ yields two distinct possibilities. Thus, the image of one point in the $z$-plane is two points in the $w$-plane. If we define $w = u + iv$, the component functions $u[x, y]$ and $v[x, y]$ can be obtained by solving the equations

$$x == u^2 - v^2 \quad y == 2uv \tag{1.34}$$

Substituting $v \to y/2u$ and solving the quadratic equation for $u^2$, we find

$$4u^4 - 4u^2 x - y^2 == 0 \Longrightarrow u^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2} \Longrightarrow u = \pm\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} \tag{1.35}$$

where the positive root is required in $u^2$ to ensure real $u$. Then, solving for $v$ and rationalizing the expression under the square root, we obtain

$$v = \frac{y}{2u} = \pm\frac{y}{2}\sqrt{\frac{2}{\sqrt{x^2 + y^2} + x}} = \pm y\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2y^2}} \tag{1.36}$$

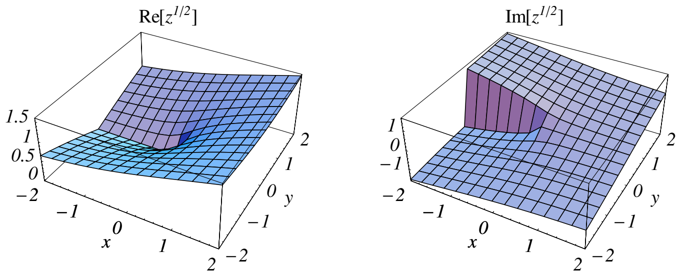


**Figure 1.7.** Real and imaginary components of $z^{1/2}$.

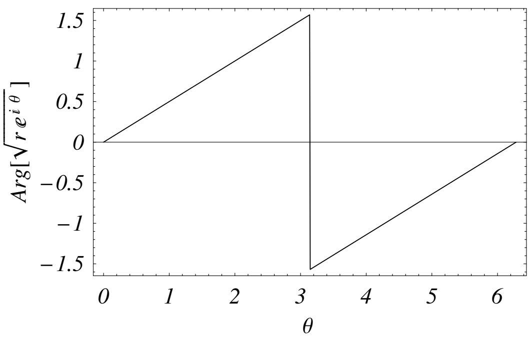


**Figure 1.8.** Dependence of Arg$[z^{1/2}]$ upon polar angle.

Finally, taking the positive root of $y^2$, we obtain

$$u = \pm\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \qquad v = \pm\,\mathrm{Sign}[y]\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \tag{1.37}$$

where the relative sign between $u$ and $v$ is determined by the sign of $y$. Note that there are only two, not four, solutions. The principal branches of the component functions are plotted in Fig. 1.7, where it is customary, though arbitrary, to select the positive branch of $u$ so that positive square roots are obtained on the positive real axis. Similar figures are obtained for other positive nonintegral powers, rational or irrational.

Notice that $v = \mathrm{Im}[\sqrt{z}]$ is discontinuous on the negative real axis. The real part is continuous, but its derivative with respect to $y$ is discontinuous on the negative real axis. Consider the image of a circular path $z = re^{i\theta}$, $0 \le \theta \le 2\pi$ under the mapping $w = \sqrt{z}$. The argument of $w$ changes abruptly from $\pi$ to $-\pi$ as the negative real axis is crossed from above, as sketched in Fig. 1.8.

In order to define a well-behaved monovalent function, we must include in the definition of $f$ a rule for selecting the appropriate output value when the mapping $z \to w$ is multivalent. The customary solution is to introduce a branch cut along the negative real axis by restricting the range of the argument of $z$ to $-\pi < \theta \leq \pi$ and agreeing not to cross the cut in the $z$-plane. Thus, Sqrt and Arg employ the same branch cut, shown in Fig. 1.5. We imagine that the cut is infinitesimally below the negative real axis so that the argument of negative real numbers is $\pi$ and $x < 0 \Longrightarrow \sqrt{x} = i\sqrt{|x|}$. The discontinuity in $\text{Arg}[z^\alpha]$ across the branch cut depends upon $\alpha$. The end points of the branch cut are known as *branch points* at which discontinuities first open. Here, the most important branch point is at $z = 0$, but one often says that there is also a branch point at $\infty$. This somewhat sloppy language means that for large $|z|$ the branch cut is parametrized by $z = Re^{i\theta}$ with $R \to \infty$, but the choice of $\theta$ remains arbitrary; here we happened to choose $\theta = \pi$.

Next consider the slightly more complicated function

$$f[z] = \left(z^2 - 1\right)^{1/2} = \left((z-1)(z+1)\right)^{1/2} \tag{1.38}$$

Our experience with the square root suggests that we must pay close attention to the points $z = \pm 1$ that are the branch points for $(z \mp 1)^{1/2}$. By factoring the argument of the square root and choosing ranges for the phase of each factor according to

$$z_1 = z - 1 = r_1 e^{i\theta_1}, \quad -\pi < \theta_1 \leq \pi \tag{1.39}$$

$$z_2 = z + 1 = r_2 e^{i\theta_2}, \quad -\pi < \theta_2 \leq \pi \tag{1.40}$$

we obtain a single-valued version defined by

$$f_1[z] = \sqrt{r_1 r_2}\,\text{Exp}\left[i\left(\frac{\theta_1 + \theta_2}{2} + k\pi\right)\right], \quad k \in \{0, 1\} \tag{1.41}$$

and the branch cuts indicated in Fig. 1.9. The principal branch is defined, somewhat arbitrarily, by $k = 0$ because that gives a positive root for $z$ on the real axis with $x > 1$. The two heavy points show the branch points in the two factors $(z \pm 1)^{1/2}$ and the lines anchored by those points show the associated branch cuts. Here we decided to draw both branch cuts to the left, as is customary for $\text{Arg}[z]$ or $z^{1/2}$, such that both polar angles are defined in the range $-\pi < \theta_{1,2} \leq \pi$. Since it is clear that any discontinuities will be found along the real axis where the two phases may be discontinuous, the structure of the function can be investigated using strategically chosen points on both sides of the real axis labeled *a*–*f* in the figure and its accompanying Table 1.1. This table shows that $f_1[z]$ is discontinuous across the portion of real axis between the two branch points, namely $-1 < x < 1$, but is continuous elsewhere. In effect, the overlapping branch cuts cancel each other in the region $x < -1$ because, at least for this function, the discontinuity is simply a sign change; however, the behavior of overlapping cuts is not always this simple.

Alternatively, if we define the phases according to

$$z_1 = z - 1 = r_1 e^{i\theta_1}, \quad 0 \leq \theta_1 < 2\pi \tag{1.42}$$

$$z_2 = z + 1 = r_2 e^{i\theta_2}, \quad -\pi < \theta_2 \leq \pi \tag{1.43}$$

we obtain another version

$$f_2[z] = \sqrt{r_1 r_2}\,\text{Exp}\left[i\left(\frac{\theta_1 + \theta_2}{2} + k\pi\right)\right], \quad k \in \{0, 1\} \tag{1.44}$$

in which the cut in $z_1^{1/2}$ is now directed toward the right while the cut in $z_2^{1/2}$ remains to the left; these cuts are illustrated in Fig. 1.10. The same selection of trial points now produces

**Table 1.1.** Selected values of $f_1[z]$ defined by Eq. (1.41).

| Point | $x$ | $y$ | $\theta_1$ | $\theta_2$ | $f_1[z]$ |
|---|---|---|---|---|---|
| $a$ | $x > 1$ | $\varepsilon$ | 0 | 0 | $\sqrt{x^2 - 1}$ |
| $b$ | $x > 1$ | $-\varepsilon$ | 0 | 0 | $\sqrt{x^2 - 1}$ |
| $c$ | $-1 < x < 1$ | $\varepsilon$ | $\pi$ | 0 | $i\sqrt{1 - x^2}$ |
| $d$ | $-1 < x < 1$ | $-\varepsilon$ | $-\pi$ | 0 | $-i\sqrt{1 - x^2}$ |
| $e$ | $x < -1$ | $\varepsilon$ | $\pi$ | $\pi$ | $-\sqrt{x^2 - 1}$ |
| $f$ | $x < -1$ | $-\varepsilon$ | $-\pi$ | $-\pi$ | $-\sqrt{x^2 - 1}$ |

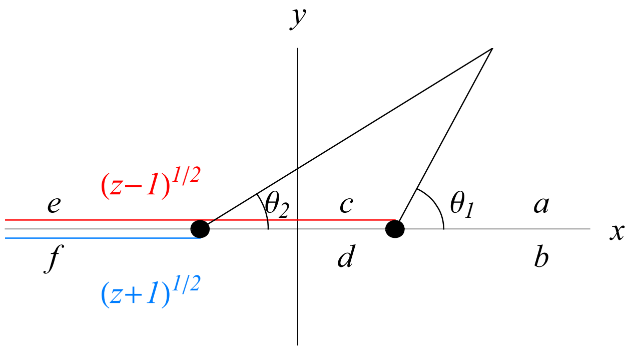


**Figure 1.9.** Branch cuts for $f_1[z]$ defined by Eq. (1.41).

Table 1.2 that shows that $f_2[z]$ is continuous for $-1 < x < 1$ but is discontinuous everywhere else on the real axis. Although their algebraic definitions are the same, $f_1[z]$ and $f_2[z]$ are clearly different functions, fraternal twins that are distinguished by their branch cuts. The choice of cuts is a fundamental aspect of the definition of a single-valued version of an inherently multivalued function; the definition is not complete until the cuts are specified. For any particular application the most appropriate version may depend upon other aspects of the problem, such as physical boundary conditions, or may be chosen for convenience. If one is most interested in small values of $|z|$ it will probably be more convenient to choose $f_2[z]$, but for large values $f_1[z]$ is probably preferable, but consistency must be maintained through any particular problem. Furthermore, although the two versions suggested here are the most common, they do not exhaust the possibilities.

The moral of this somewhat belabored exercise is that one must be very careful in manipulating expressions involving complex variables and avoid making unintentional assumptions about the phases of various subexpressions. Most people tend to be very careless with phases, automatically replacing $\sqrt{s^2}$ by $s$ without knowing whether s is positive or negative or complex. Similarly, many people complain that *MATHEMATICA*® often does not perform simplifications that are perceived to be obvious. The reason for this is that *MATHEMATICA*® hates to make mistakes and will not make unjustified assumptions about

**Table 1.2.** Selected values of $f_2[z]$ defined by Eq. (1.44).

| Point | $x$ | $y$ | $\theta_1$ | $\theta_2$ | $f_2[z]$ |
|---|---|---|---|---|---|
| $a$ | $x > 1$ | $\varepsilon$ | $0$ | $0$ | $\sqrt{x^2-1}$ |
| $b$ | $x > 1$ | $-\varepsilon$ | $2\pi$ | $0$ | $-\sqrt{x^2-1}$ |
| $c$ | $-1 < x < 1$ | $\varepsilon$ | $\pi$ | $0$ | $i\sqrt{1-x^2}$ |
| $d$ | $-1 < x < 1$ | $-\varepsilon$ | $\pi$ | $0$ | $i\sqrt{1-x^2}$ |
| $e$ | $x < -1$ | $\varepsilon$ | $\pi$ | $\pi$ | $-\sqrt{x^2-1}$ |
| $f$ | $x < -1$ | $-\varepsilon$ | $\pi$ | $-\pi$ | $\sqrt{x^2-1}$ |

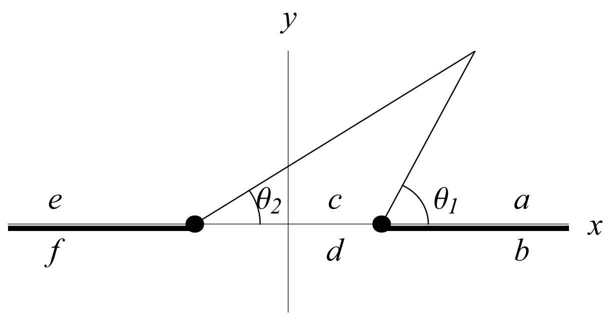


**Figure 1.10.** Branch cuts for $f_2[z]$ defined by Eq. (1.44).

whether a variable is real or, if it is, about its sign – it assumes that all variables are complex unless told otherwise. The Simplify function has an option that permits the user to specify permissible assumptions, such as one variable is real, another positive, a third a negative integer, etc. When you take responsibility for these assumptions, MATHEMATICA® will usually go much further in simplifying your expressions. Often it still will not reach the elegant representation that one might find in a textbook, but its manipulations will be correct and that is what matters most.

## 1.3 Functions as Mappings

A function $f$ maps the complex variable $z = (x, y)$ into a complex *image* $w = (u, v)$ according to rules specified in the definition $w = f[z]$. Thus, $f$ maps points in the complex $z$-plane onto points in the complex $w$-plane, a mapping of $\mathbb{C} \to \mathbb{C}$. For a single-valued function the image of a point is a point, but for a multiple-valued function the image of a point may be a set of points. For continuous functions, the image of a line segment (arc) in the input plane will be one or more arcs in the output plane. Often considerable insight into the properties of a function may be obtained by examining the images of *coordinate lines* (lines of constant $x$ or constant $y$). Below we analyze the mappings produced by some familiar functions.

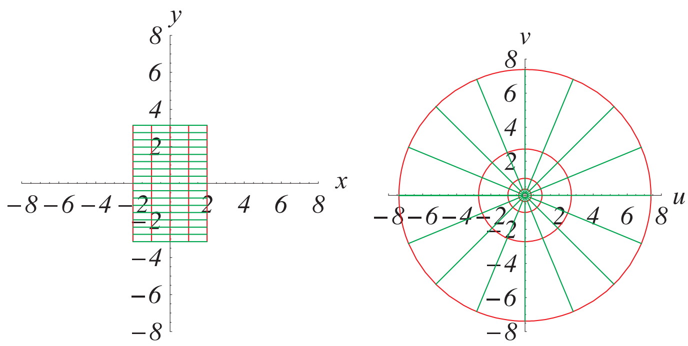


**Figure 1.11.** Mapping: $u + iv = e^{x+iy}$. Lines of constant $y$ are mapped onto radial lines while lines lines of constant $x$ are mapped onto circles in the $w$-plane.

### 1.3.1 Mapping: $w = e^z$

The exponential function is defined by the mapping

$$w = e^z = e^x\big(\mathrm{Cos}[y] + i\,\mathrm{Sin}[y]\big) = u + iv \Longrightarrow u[x, y] = e^x\,\mathrm{Cos}[y] \quad v[x, y] = e^x\,\mathrm{Sin}[y] \tag{1.45}$$

The image of a grid of coordinate lines is sketched in Fig. 1.11. Lines of constant $y$ are mapped into radial lines, while lines of constant $x$ are mapped into circles. The origin, with $x = 0$, is mapped onto the unit circle. Increasingly positive $x = x_0 > 0$, mapped onto circles of exponentially increasing radius $e^{x_0}$, and increasingly negative $x = x_0 < 0$ mapped onto exponentially tighter circles, practically indiscernible in this figure. Thus, the images of the coordinate lines remain orthogonal, but the mapping severely distorts distances.

It is important to recognize that the mapping produced by the exponential function is *many-to-one* because

$$\mathrm{Exp}[z + 2\pi i k] = \mathrm{Exp}[z] \quad \text{for integer } k \tag{1.46}$$

is periodic, so that infinitely many input points $z = z_0 + 2\pi i k$ are mapped onto the same image point. Thus, any strip $|y - y_0| \leq \pi$ in the $z$-plane is mapped onto the entire $w$-plane and neighboring strips would replicate the covering of the $w$-plane. Consequently, the inverse function $z = \log[w]$ is many-valued because it represents a *one-to-many* mapping. By convention, we define the principal branch of the logarithm function

$$\mathrm{Log}[z] = \mathrm{Log}[|z|] + i\,\mathrm{Arg}[z] \quad \text{with} \quad -\pi < \mathrm{Arg}[z] \leq \pi \tag{1.47}$$

by limiting the phase $y = \mathrm{Arg}[w]$ to the strip $-\pi < y \leq \pi$ by means of a branch cut along the negative real axis, as indicated by the thick line in Fig. 1.12. With this convention one obtains the following principal values:

$$\mathrm{Log}[1] = 0 \quad \mathrm{Log}[i] = \frac{i\pi}{2} \quad \mathrm{Log}[-1] = i\pi \quad \mathrm{Log}[-i] = -\frac{i\pi}{2} \tag{1.48}$$

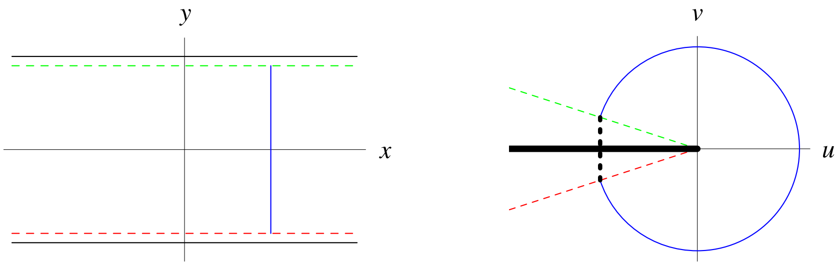


**Figure 1.12.** Mapping: $u + iv = \text{Exp}[x + iy]$. The mapping of any strip $|y - y_0| \leq \pi$ covers the $w$-plane. The branch cut for the inverse mapping is shown in the $w$-plane.

One might imagine the $w$-plane as a circular paper disk with a cut from its edge to the center, which prevents a continuous curve to be drawn crossing the cut. A vertical line segment in the $z$-plane between $y = -\pi + \varepsilon$ and $y = \pi - \varepsilon$ is mapped onto a circular arc between $\text{Arg}[w] = -\pi + \varepsilon$ and $\text{Arg}[w] = \pi - \varepsilon$. Although the two ends of the arc approach other from opposite sides of the branch cut as $\varepsilon \to 0^+$, they differ in phase by $2\pi$. Therefore, although the branch cut permits the inverse function $z = \text{Log}[w]$ to be defined as a single-valued mapping $w \to z$, that function is discontinuous across the branch cut. Often single-valuedness comes only at the expense of discontinuities.

Neighboring strips $y_n = y_0 + n\pi$ simply remap the entire $w$-plane. One might imagine an infinite collection of $w$-planes, called *Riemann sheets*, stacked on top of each other such that curves which cross the branch cut move from one Riemann sheet to the next. The index $n$ then identifies a particular Riemann sheet, with the principal branch represented by $n = 0$. Thus, it is useful to distinguish between a multivalued log and a single-valued Log defined by

$$\begin{aligned} w == e^z \Longrightarrow z &= \log[w] = \log[|w|] + i \arg[w] \\ &= \log[|w|] + i\,\text{Arg}[w] + 2\pi i n = \text{Log}[w] + 2\pi i n \end{aligned} \tag{1.49}$$

As a curve winds around the origin of the $w$-plane in a counterclockwise sense, the argument increases continuously and each time one crosses the branch cut one moves from one sheet to the next and increments the *winding number* $n$ by one unit. Clockwise winding decrements $n$, which is permitted to be negative also. Furthermore, the choice $y_0 = 0$ is not unique and other choices would rotate the branch cut in the $w$-plane. The single-valued function produced by the most common choice of branch cut is described as the *principal branch*, but for some problems it may become convenient to make a different choice. However, the branch cuts used for Arg, Log, and related functions are correlated and must be chosen consistently throughout a particular calculation.

Physics calculations normally must produce a unique answer that can be compared with a measurable quantity, such that physical functions must be based upon single-valued functions. Similarly, if one is to compute the value of an expression using a computer program, there must be a unique result. Numerical methods cannot tolerate multivalued expressions – the programmer must provide an unambiguous prescription for selecting the appropriate branches of multivalued functions; a machine cannot perform that job for you. It is useful to visualize a function as a machine. When you supply appropriate input, it produces a definite and predictable output. A function is not really defined until its branch cuts and its discontinuities across those cuts are completely specified. Furthermore, there is often considerable flexibility in the selection of cuts that can be exploited to simplify the problem at hand, one selection for one problem and another for the next. Therefore, one must always be aware of the branch cuts used to regularize an inherently multivalued function.

### 1.3.2 Mapping: $w = \mathrm{Sin}[z]$

The sine function is extended to complex variables by the definition

$$\mathrm{Sin}[z] = \frac{e^{iz} - e^{-iz}}{2i} \tag{1.50}$$

Using

$$\begin{aligned} z = x + iy \Longrightarrow \mathrm{Sin}[z] &= \frac{e^{-y}\left(\mathrm{Cos}[x] + i\,\mathrm{Sin}[x]\right) - e^{y}\left(\mathrm{Cos}[x] - i\,\mathrm{Sin}[x]\right)}{2i} \\ &= \frac{e^{y} + e^{-y}}{2}\,\mathrm{Sin}[x] + i\frac{e^{y} - e^{-y}}{2}\,\mathrm{Cos}[x] \end{aligned} \tag{1.51}$$

and the familiar definitions

$$\mathrm{Cosh}[y] = \frac{e^{y} + e^{-y}}{2} \qquad \mathrm{Sinh}[y] = \frac{e^{y} - e^{-y}}{2} \tag{1.52}$$

for real variables, the components of the sine function become

$$u + iv = \mathrm{Sin}[x + iy] \Longrightarrow u = \mathrm{Sin}[x]\,\mathrm{Cosh}[y] \quad v = \mathrm{Cos}[x]\,\mathrm{Sinh}[y] \tag{1.53}$$

The mapping of $(x, y)$ coordinate lines is illustrated in Fig. 1.13. Lines of constant $x$ are mapped into hyperbolae while lines of constant $y$ are mapped into confocal ellipses with foci at $(u, v) = (\pm 1, 0)$. The definition of the inverse mapping $z = \mathrm{ArcSin}[w]$ requires branch cuts because any strip $|x - x_0| \leq \pi$ is mapped onto the entire $w$-plane. It is customary to map the principal branch of ArcSin onto the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$, but two choices remain for the branch cuts. Recognizing that as $|x| \to \frac{\pi}{2}$ the hyperbolic images of the vertical lines collapse upon the real axis, the most common choice is to place the branch cuts along the open interval $|u| > 1$. This choice reduces to the standard definition of $\mathrm{ArcSin}[x]$ for real arguments in the range $|x| \leq 1$.

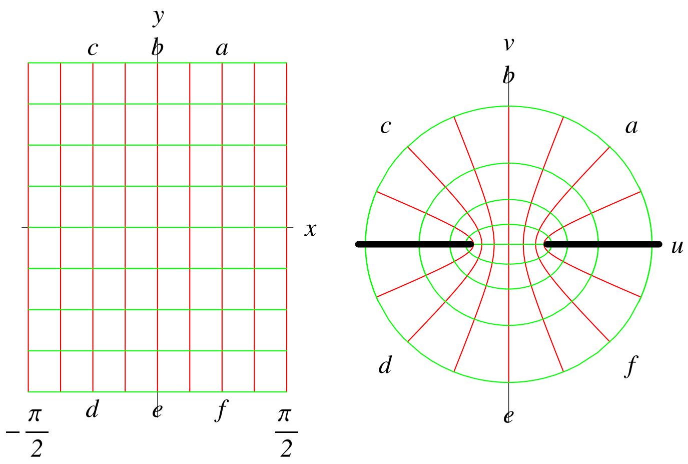


**Figure 1.13.** Mapping: $u + iv = \mathrm{Sin}[x + iy]$. Lines of constant $x$ are mapped onto hyperbolas while lines of constant $y$ are mapped onto confocal ellipses.

## 1.4 Elementary Functions and Their Inverses

### 1.4.1 Exponential and Logarithm

Some properties of the exponential are preserved by extension from the real axis to the complex plane. For example, using

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1} e^{x_2} \left(\mathrm{Cos}[y_1] + i\,\mathrm{Sin}[y_1]\right)\left(\mathrm{Cos}[y_2] + i\,\mathrm{Sin}[y_2]\right) \\ &= e^{x_1+x_2} \left(\mathrm{Cos}[y_1]\,\mathrm{Cos}[y_2] - \mathrm{Sin}[y_1]\,\mathrm{Sin}[y_2]\right. \\ &\quad \left. + i\left(\mathrm{Sin}[y_1]\,\mathrm{Cos}[y_2] + \mathrm{Cos}[y_1]\,\mathrm{Sin}[y_2]\right)\right) \\ &= e^{x_1+x_2} \left(\mathrm{Cos}[y_1 + y_2] + i\,\mathrm{Sin}[y_1 + y_2]\right) \end{aligned} \tag{1.54}$$

we find

$$e^{z_1} e^{z_2} = e^{z_1+z_2} \tag{1.55}$$

Similarly one can easily prove

$$\frac{1}{e^z} = e^{-z} \qquad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2} \tag{1.56}$$

and

$$(e^z)^n = e^{nz} \quad \text{for} \quad n \in \text{ Integers} \tag{1.57}$$

for integer $n$. However, one must generally assume that

$$(e^{z_1})^{z_2} \neq e^{z_1 z_2} \tag{1.58}$$

for arbitrary powers $z_2$. For example, $(e^z)^{1/n}$ is multivalued, producing $n$ values for integer $n$, while $e^{z/n}$ represents a unique complex number.

We define the multivalent logarithm function $w = \log[z]$ in terms of the solutions to the equation $z == e^w$, such that

$$\begin{aligned} &z == e^w, \\ &z = |z| e^{i\arg[z]} \Longrightarrow \log[z] = \log[|z|] + i\arg[z] = \mathrm{Log}[|z|] + i\,\mathrm{Arg}[z] + 2\pi i n \end{aligned} \tag{1.59}$$

where the principal value is used for the logarithm of the positive real number $|z|$ and where $n$ is an arbitrary integer. Thus, this version of the logarithm function produces infinitely many values for any $z$ . Consequently, we cannot simply replace $\log[e^z]$ by $z$ during calculations because

$$\begin{aligned} z = x + iy \Longrightarrow e^z = e^x e^{iy} \Longrightarrow \log[e^z] &= \mathrm{Log}[|e^z|] + i\,\mathrm{Arg}[e^z] + 2\pi i n \\ &= x + i(y + 2\pi n) \\ &= z + 2\pi i n \end{aligned} \tag{1.60}$$

is ambiguous. By selecting $n \to 0$, we define the single-valued principal branch as

$$\mathrm{Log}[z] = \mathrm{Log}[|z|] + i\,\mathrm{Arg}[z] \tag{1.61}$$

and obtain

$$\mathrm{Log}[e^z] = z \tag{1.62}$$

as expected. On the other hand, some functional relationships that pertain to real arguments remain true for the multivalent version but are not necessarily true for the principal branch. For example, from

$$\begin{aligned} \log[z_1 z_2] &= \log[|z_1 z_2|] + i\arg[z_1 z_2] \\ &= \log[|z_1||z_2|] + i\big(\arg[z_1] + \arg[z_2]\big) \\ &= \log[|z_1|] + \log[|z_2|] + i\big(\arg[z_1] + \arg[z_2]\big) \end{aligned} \tag{1.63}$$

we find

$$\log[z_1 z_2] = \log[z_1] + \log[z_2] \tag{1.64}$$

but the corresponding relationship for the principal branch

$$\mathrm{Log}[z_1 z_2] = \mathrm{Log}[z_1] + \mathrm{Log}[z_2] + 2\pi i n \tag{1.65}$$

is more complicated because we must deduce the appropriate $n$ from the phases of $z_1$ and $z_2$.

### 1.4.2 Powers

Powers of a complex number are defined by

$$z^\alpha = \mathrm{Exp}\big[\alpha \log[z]\big] = \mathrm{Exp}\big[\alpha\,\mathrm{Log}[|z|]\big]\,\mathrm{Exp}\big[i\alpha\arg[z]\big] \tag{1.66}$$

and are generally multivalued. In polar notation we may write

$$z = re^{i\theta} \Longrightarrow z^{\alpha} = \mathrm{Exp}\big[\alpha\,\mathrm{Log}[r]\big]\,\mathrm{Exp}\big[i\alpha(\theta + 2\pi n)\big] \tag{1.67}$$

where $n$ is an integer. This definition conforms to simple expectations for rational exponents and preserves the algebraic relationships

$$\left(z^{\alpha}\right)^{\beta} = \mathrm{Exp}\big[\beta\log[z^{\alpha}]\big] = \mathrm{Exp}\big[\beta\alpha\log[z]\big] = z^{\alpha\beta} \tag{1.68}$$

$$\left(z_1 z_2\right)^{\alpha} = \mathrm{Exp}\big[\alpha\log[z_1 z_2]\big] = \mathrm{Exp}\big[\alpha(\log[z_1] + \log[z_2])\big] = z_1^{\alpha} z_2^{\alpha} \tag{1.69}$$

However, these algebraic relationships are generally multivalent. If we use a branch cut for the logarithm under the negative real axis, the same branch cut must be used for multivalent powers that are complex, nonintegral, or have negative real parts. The discontinuity in the argument of $z^{\alpha}$ across the branch cut is then

$$\Delta\,\mathrm{Arg}[z^{\alpha}] = \mathrm{Limit}[\mathrm{Arg}[x + i\varepsilon] - \mathrm{Arg}[x - i\varepsilon], \varepsilon \to 0] = \mathrm{Mod}[2\pi\alpha, 2\pi] \tag{1.70}$$

If $\alpha$ is rational there are a finite number of Riemann sheets, but for irrational $\alpha$ there are an infinite number of Riemann sheets. The principal branch is given by

$$\text{principal branch: } z^{\alpha} = \mathrm{Exp}\big[\alpha\,\mathrm{Log}[z]\big] = \mathrm{Exp}\big[\alpha\,\mathrm{Log}[|z|]\big]\,\mathrm{Exp}\big[i\alpha\,\mathrm{Arg}[z]\big] \tag{1.71}$$

### 1.4.3 Trigonometric and Hyperbolic Functions

Recognizing that

$$\mathrm{Cos}[x] = \frac{e^{ix} + e^{-ix}}{2} \qquad \mathrm{Sin}[x] = \frac{e^{ix} - e^{-ix}}{2i} \qquad \mathrm{Tan}[x] = \frac{\mathrm{Sin}[x]}{\mathrm{Cos}[x]} \tag{1.72}$$

$$\mathrm{Cosh}[x] = \frac{e^{x} + e^{-x}}{2} \qquad \mathrm{Sinh}[x] = \frac{e^{x} - e^{-x}}{2} \qquad \mathrm{Tanh}[x] = \frac{\mathrm{Sinh}[x]}{\mathrm{Cosh}[x]} \tag{1.73}$$

for $x \in \mathbb{R}$, we define

$$\mathrm{Cos}[z] = \frac{e^{iz} + e^{-iz}}{2} \qquad \mathrm{Sin}[z] = \frac{e^{iz} - e^{-iz}}{2i} \qquad \mathrm{Tan}[z] = \frac{\mathrm{Sin}[z]}{\mathrm{Cos}[z]} \tag{1.74}$$

$$\mathrm{Cosh}[z] = \frac{e^{z} + e^{-z}}{2} \qquad \mathrm{Sinh}[z] = \frac{e^{z} - e^{-z}}{2} \qquad \mathrm{Tanh}[z] = \frac{\mathrm{Sinh}[z]}{\mathrm{Cosh}[z]} \tag{1.75}$$

for $z \in \mathbb{C}$. Inverting these expressions gives

$$e^{iz} = \mathrm{Cos}[z] + i\,\mathrm{Sin}[z] \qquad e^{-iz} = \mathrm{Cos}[z] - i\,\mathrm{Sin}[z] \tag{1.76}$$

$$e^{z} = \mathrm{Cosh}[z] + \mathrm{Sinh}[z] \qquad e^{-z} = \mathrm{Cosh}[z] - \mathrm{Sinh}[z] \tag{1.77}$$

Obviously,

$$\mathrm{Cos}[-z] = \mathrm{Cos}[z] \qquad \mathrm{Sin}[-z] = -\mathrm{Sin}[z] \qquad \mathrm{Tan}[z] = -\mathrm{Tan}[z] \tag{1.78}$$

$$\mathrm{Cosh}[-z] = \mathrm{Cosh}[z] \qquad \mathrm{Sinh}[-z] = -\mathrm{Sinh}[z] \qquad \mathrm{Tanh}[z] = -\mathrm{Tanh}[z] \tag{1.79}$$

and

$$\mathrm{Cos}[iz] = \mathrm{Cosh}[z] \qquad \mathrm{Sin}[iz] = i\,\mathrm{Sinh}[z] \qquad \mathrm{Tan}[iz] = i\,\mathrm{Tanh}[z] \tag{1.80}$$

$$\mathrm{Cosh}[iz] = \mathrm{Cos}[z] \qquad \mathrm{Sinh}[iz] = i\,\mathrm{Sin}[z] \qquad \mathrm{Tanh}[iz] = i\,\mathrm{Tan}[z] \tag{1.81}$$

Multiplying out the expressions

$$\begin{aligned}\mathrm{Exp}\big[i(z_1+z_2)\big] &= \mathrm{Cos}[z_1+z_2] + i\,\mathrm{Sin}[z_1+z_2]\\ &= \big(\mathrm{Cos}[z_1] + i\,\mathrm{Sin}[z_1]\big)\big(\mathrm{Cos}[z_2] + i\,\mathrm{Sin}[z_2]\big)\\ &= e^{iz_1}e^{iz_2}\end{aligned} \tag{1.82}$$

$$\begin{aligned}\mathrm{Exp}\big[i(z_1-z_2)\big] &= \mathrm{Cos}[z_1-z_2] + i\,\mathrm{Sin}[z_1-z_2]\\ &= \big(\mathrm{Cos}[z_1] + i\,\mathrm{Sin}[z_1]\big)\big(\mathrm{Cos}[z_2] - i\,\mathrm{Sin}[z_2]\big)\\ &= e^{iz_1}e^{-iz_2}\end{aligned} \tag{1.83}$$

one quickly deduces the addition formulae

$$\mathrm{Cos}[z_1+z_2] = \mathrm{Cos}[z_1]\,\mathrm{Cos}[z_2] - \mathrm{Sin}[z_1]\,\mathrm{Sin}[z_2] \tag{1.84}$$

$$\mathrm{Cosh}[z_1+z_2] = \mathrm{Cosh}[z_1]\,\mathrm{Cosh}[z_2] + \mathrm{Sinh}[z_1]\,\mathrm{Sinh}[z_2] \tag{1.85}$$

$$\mathrm{Sin}[z_1+z_2] = \mathrm{Sin}[z_1]\,\mathrm{Cos}[z_2] + \mathrm{Cos}[z_1]\,\mathrm{Sin}[z_2] \tag{1.86}$$

$$\mathrm{Sinh}[z_1+z_2] = \mathrm{Sinh}[z_1]\,\mathrm{Cosh}[z_2] + \mathrm{Cosh}[z_1]\,\mathrm{Sinh}[z_2] \tag{1.87}$$

and

$$\mathrm{Cos}[z]^2 + \mathrm{Sin}[z]^2 = 1 \qquad \mathrm{Cosh}[z]^2 - \mathrm{Sinh}[z]^2 = 1 \tag{1.88}$$

Combining these results, we obtain

$$\mathrm{Cos}[x+iy] = \mathrm{Cos}[x]\,\mathrm{Cosh}[y] - i\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y] \tag{1.89}$$

$$\mathrm{Cosh}[x+iy] = \mathrm{Cosh}[x]\,\mathrm{Cos}[y] + i\,\mathrm{Sinh}[x]\,\mathrm{Sin}[y] \tag{1.90}$$

$$\mathrm{Sin}[x+iy] = \mathrm{Sin}[x]\,\mathrm{Cosh}[y] + i\,\mathrm{Cos}[x]\,\mathrm{Sinh}[y] \tag{1.91}$$

$$\mathrm{Sinh}[x+iy] = \mathrm{Sinh}[x]\,\mathrm{Cos}[y] + i\,\mathrm{Cosh}[x]\,\mathrm{Sin}[y] \tag{1.92}$$

for $\{x, y\} \in \mathbb{R}$. Expressions for the real and imaginary components of inverse trigonometric functions are developed in the exercises.

### 1.4.4 Standard Branch Cuts

Although there is often some flexibility in the choice of branch cuts, the cuts for related functions are correlated. Table 1.3 lists the standard choices for elementary functions, but other choices can facilitate certain calculations. Parentheses (square brackets) indicate an open (closed) interval.

**Table 1.3.** Standard definitions for principal branch of elementary functions.

| Function | Branch cuts |
|---|---|
| Abs | none |
| Arg | $(-\infty, 0)$ |
| Sqrt | $(-\infty, 0)$ |
| $z^s$, nonintegral $s$ with $\mathrm{Re}[s] > 0$ | $(-\infty, 0)$ |
| $z^s$, nonintegral $s$ with $\mathrm{Re}[s] \leq 0$ | $(-\infty, 0]$ |
| Exp | none |
| Log | $(-\infty, 0]$ |
| trigonometric functions | none |
| ArcSin, ArcCos | $(-\infty, -1)$ and $(1, \infty)$ |
| ArcTan | $(-i\infty, -i]$ and $[i, i\infty)$ |
| ArcCsc and ArcSec | $(-1, 1)$ |
| ArcCot | $[-i, i]$ |
| hyperbolic functions | none |
| ArcSinh | $(-i\infty, -i)$ and $(i, i\infty)$ |
| ArcCosh | $(-\infty, 1)$ |
| ArcTanh | $(-i\infty, -i]$ and $[i, i\infty)$ |
| ArcCsch | $(-i, i)$ |
| ArcSech | $(-\infty, 0]$ and $(1, \infty)$ |
| ArcCoth | $[-1, 1]$ |

## 1.5 Sets, Curves, Regions and Domains

The basic concept used to characterize sets, curves, and regions in the complex plane is *neighborhood*. A neighborhood of $z_0$ consists of the set of all points that satisfy the inequality $|z - z_0| < \varepsilon$; the radius $\varepsilon$ is usually assumed to be small. A point $z$ is an *interior point* of the set $S$ if there exists a neighborhood containing only points belonging to $S$. Conversely, a point is *exterior* to $S$ if there exists a neighborhood that does not contain any points belonging to $S$. Finally, a *boundary point* is neither interior nor exterior to $S$ because any neighborhood, no matter how small, contains both points which belong to $S$ and points which do not. An *open set* is a set for which every point is an interior point; in other words, an open set contains none of its boundary points. A *closed set*, on the other hand, contains all of its boundary points. The *closure* of $S$ consists of $S$ plus all of its boundary points and is denoted $\bar{S}$. Note that some sets, such as $0 < |z| \leq 1$, are neither open nor closed because they contain some but not all of their boundary points, while $\mathbb{C}$ is both open and closed because there are no boundary points. A set is *bounded* if all points lie within a disk $|z| < R$ for some finite $R$ and is *unbounded* otherwise. Finally, a point $z_0$ is an *accumulation point* of $S$ if every neighborhood contains at least one other point that also belongs to $S$. Thus, a closed set contains all of its accumulation points and, conversely, any set which contains all of its accumulation points is closed. For example, the origin is the only accumulation point of the set $\{z_n = \frac{1}{n}, n = 1, \infty\}$.

Any set of points that consists only of boundary points constitutes a *curve*. For example, the set of points that satisfy the equation $|z - z_0| == R$ describes a circle of radius $R$

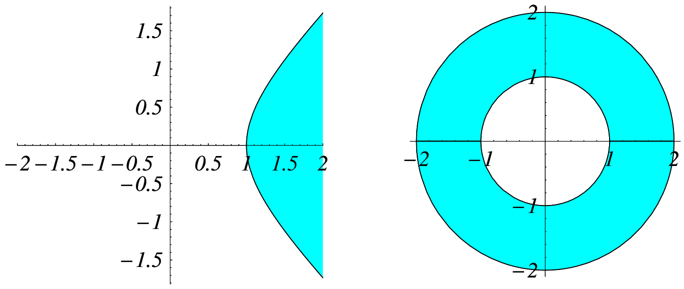


**Figure 1.14.** Left: A simply connected domain; right: a mutiply connected domain.

centered on $(x_0, y_0)$ and is an example of a curve. An *arc* is a curve described by the parametric equation $z = (x[t], y[t])$ where $x[t]$ and $y[t]$ are continuous real functions of the real variable $t_{\min} \le t \le t_{\max}$. An arc is *simple* if it does not intersect itself, in other words if $t_1 \ne t_2 \Longrightarrow z[t_1] \ne z[t_2]$ for $t_{\min} < t_1, t_2 < t_{\max}$. A *simple closed curve* does not intersect itself except at the endpoints where $z[t_{\min}] = z[t_{\max}]$.

An open set is *connected* if any pair of points can be joined by a polygonal path that lies entirely within the set. An open connected set is called a *domain*. For example, the annulus $1 < |z| < 2$ is a domain because it is open and connected. Any neighborhood is also a domain. A domain $D$ is described as *simply connected* if all simple closed curves within $D$ enclose only points that are also within $D$ and is described as multiply connected otherwise. A domain together with a subset of its boundary points (none, some, or all) is called a *region*. For example, $\{z \ni \text{Re}[z^2] > 1 \wedge \text{Re}[z] > 0\}$ describes a simply connected domain while $\{z \ni 1 \le |z| \le 2\}$ describes a multiply connected region.

## 1.6 Limits and Continuity

The limit of $f[z]$ as $z \to z_0$ is defined to be the complex number $w_0$ if for each arbitrarily small positive number $\varepsilon$ there exists a positive number $\delta$ for which $0 < |f[z] - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. Geometrically, this definition requires that the image $w = f[z]$ for any point $z$ in a $\delta$-neighborhood of $z_0$, with the possible exception of $z_0$ itself, should lie within an $\varepsilon$-neighborhood of $w_0$. Note that this definition requires all points in the neighborhood of $z_0$ to be mapped within the neighborhood of $w_0$ but does not require the mapping to constitute a domain because the mapping need not produce a connected set. Furthermore, the limit $z \to z_0$ may be approached in an arbitrary manner. However, the present definition does not apply to points $z_0$ which lie on the boundary of the domain on which $f[z]$ is defined because in that case the $\delta$-neighborhood contains points at which $f[z]$ may be undefined. Nevertheless, we can extend the definition of limit by limiting the requirements on the inequalities to those points in the neighborhood of $z_0$ that lie within the domain of $f$.

Direct application of the definition of limits can be quite cumbersome, but a few almost self-evident theorems are quite helpful.

**Theorem 1.** *Let* $f[z] = u[x, y] + iv[x, y]$, $z_0 = x_0 + iy_0$, *and* $w_0 = u_0 + iv_0$. *Then*

$$\operatorname*{Lim}_{z\to z_0} f[z] = w_0 \tag{1.93}$$

*if and only if*

$$\operatorname*{Lim}_{(x,y)\to(x_0,y_0)} u[x, y] = u_0 \quad \textit{and} \quad \operatorname*{Lim}_{(x,y)\to(x_0,y_0)} v[x, y] = v_0 \tag{1.94}$$

**Theorem 2.** *Let* $f_0 = \mathrm{Lim}_{z\to z_0} f[z]$ *and* $g_0 = \mathrm{Lim}_{z\to z_0} g[z]$. *Then*

1. $\mathrm{Lim}_{z\to z_0}(f[z] + g[z]) = f_0 + g_0$
2. $\mathrm{Lim}_{z\to z_0} f[z]g[z] = f_0 g_0$
3. $\mathrm{Lim}_{z\to z_0} \frac{f[z]}{g[z]} = \frac{f_0}{g_0}$ *if* $g_0 \neq 0$
4. $\mathrm{Lim}_{z\to z_0} \mathrm{Abs}[f[z]] = \mathrm{Abs}[f_0]$

A function $f[z]$ is *continuous* at $z_0$ if $\lim_{z\to z_0} f[z] = f[z_0]$ and is continuous in a region $R$ if it is continuous at all points within that region. Note that this definition implicitly requires $f[z]$ and its limit at $z_0$ to exist.

**Theorem 3.** *Let* $f[z]$ *be defined in a neighborhood of* $z_0$ *and suppose that for all points in that neighborhood* $f[z]$ *lies within the domain of* $g[z]$. *Then if* $f[z]$ *is continuous at* $z_0$ *and* $g[z]$ *is continuous at* $f[z_0]$, *it follows that* $g[f[z]]$ *is continuous at* $z_0$.

Consider a sequence of complex numbers $\{z_n\}$. The limit of a sequence $z_n \to w$ requires that $|z_n - w| < \varepsilon$ whenever $n > N[\varepsilon]$. A *Cauchy sequence* requires $|z_n - z_m| \to 0$ as $n, m \to \infty$. A sequence converges if and only if it is a Cauchy sequence.

## 1.7 Differentiability

### 1.7.1 Cauchy–Riemann Equations

Let $w = f[z] = u[x, y] + iv[x, y]$ be a function of the complex variable $z = x + iy$ and define its derivative by

$$f'[z] = \frac{dw}{dz} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z\to 0} \frac{f[z + \Delta z] - f[z]}{\Delta z} \tag{1.95}$$

Although this definition is simply the obvious generalization of the derivative of a real-valued function of a real variable, the higher dimensionality of complex variables imposes nontrivial requirements upon differentiable complex functions. The existence of such a derivative requires

1. $f[z]$ be defined at $z$
2. $f[z] \neq \infty$
3. the limit must be independent of the direction in which $\Delta z \to 0$.

The independence of direction is a strong condition which leads to the *Cauchy–Riemann equations*, henceforth denoted CR. Approaching the limit using variations along coordinate directions, one finds

$$\begin{aligned}\Delta z = \Delta x \Longrightarrow & \lim_{\Delta x \to 0, \Delta y = 0} \frac{u[x+\Delta x, y] + i v[x+\Delta x, y] - u[x,y] - i v[x,y]}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\end{aligned} \tag{1.96}$$

$$\begin{aligned}\Delta z = i\Delta y \Longrightarrow & \lim_{\Delta x = 0, \Delta y \to 0} \frac{u[x, y+\Delta y] + i v[x, y+\Delta y] - u[x,y] - i v[x,y]}{i\Delta y} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\end{aligned} \tag{1.97}$$

Equating the real and imaginary parts separately, then requires

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.98}$$

The CR equations are necessary but not quite sufficient to ensure differentiability. To obtain sufficient conditions, we also require continuity of the partial derivatives of component functions.

**Theorem 4.** *Let $f[z] = u[x, y] + i v[x, y]$ be defined throughout a neighborhood $|z - z_0| < \varepsilon$ and suppose that the first partial derivatives of u and v wrt x and y exist in that neighborhood and are continuous at $z_0 = (x_0, y_0)$. Then $f'[z]$ exists if those partial derivatives satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.99}$$

*Conversely, if $f'[z]$ exists, then the CR equations are satisfied.*

If $f[z]$ is differentiable at $z_0$ and throughout a neighborhood of $z_0$, then $f[z]$ is described as *analytic* (or regular or holomorphic) at $z_0$. If $f[z]$ is analytic everywhere in the finite complex plane, it is described as *entire*. Examples of entire functions include Exp, Sin, Cos, Sinh, and Cosh. Functions which are analytic except on branch cuts include Log, ArcSin, ArcCos, ArcSinh, and ArcCosh.

Recognizing that the CR equations are linear, it is trivial to demonstrate that if $f_1[z]$ and $f_2[z]$ are analytic functions in domains $D_1$ and $D_2$, then any linear combination $af_1[z] + bf_2[z]$ is also analytic in the overlapping domain $D = D_1 \cap D_2$. Similarly, it is straightforward, though tedious, to demonstrate that the product $f_1[z]f_2[z]$ also satisfies the CR equations and, hence, is analytic in $D$. Furthermore, one can show that $1/f_2[z]$ is analytic in $D_2$ where $f_2[z] \neq 0$ such that $f_1[z]/f_2[z]$ is analytic in $D$ except possibly at the zeros of the denominator. Finally, if $f_1[w]$ is analytic at $w = f_2[z]$, then $f_1[f_2[z]]$ is analytic. Formal demonstration that these familiar properties of derivatives also apply to analytic functions of a complex variable is left to the student.

**Example**

$$f[z] = z^2 \Longrightarrow u = x^2 - y^2, \quad v = 2xy \Longrightarrow \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \tag{1.100}$$

The partial derivatives are continuous throughout the complex plane and satisfy the CR equation; hence, $z^2$ is entire. In fact, one can show that any polynomial in $z$ is entire.

**Example**

$$f[z] = z^* \Longrightarrow u = x, \quad v = -y \Longrightarrow \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \tag{1.101}$$

The partial derivatives are continuous, but do not satisfy CR; hence, $z^*$ is nowhere differentiable and is not analytic anywhere. It is important to recognize that functions of a complex variable can be smooth and continuous without being differentiable. The requirements for differentiability are stricter for complex variables than for real variables because independence from direction imposes correlations between the dependencies upon the real and imaginary parts of the independent variable. Analytic functions of one complex variable are not simply functions of two real variables.

### 1.7.2 Differentiation Rules

Many of the familiar differentiation rules for real functions can be applied to complex functions. Suppose that $f[z]$ and $g[z]$ are differentiable within overlapping regions. Within the intersection of those regions, we can derive differentiation rules using the definition in terms of limits. Alternatively, by separating each function into real and imaginary components, one could also employ the CR relations.

For example, one quickly finds that the derivative of a sum

$$\begin{aligned} &F[z] = f[z] + g[z] \\ \Longrightarrow \lim_{\Delta z \to 0} \frac{F[z+\Delta z] - F[z]}{\Delta z} &= \lim_{\Delta z \to 0} \frac{f[z+\Delta z] - f[z]}{\Delta z} + \lim_{\Delta z \to 0} \frac{g[z+\Delta z] - g[z]}{\Delta z} \end{aligned} \tag{1.102}$$

reduces to the sum of derivatives

$$F[z] = f[z] + g[z] \Longrightarrow F'[z] = f'[z] + g'[z] \tag{1.103}$$

if both functions are differentiable. Similarly, the familiar rule for a differentiation of a product

$$F[z] = f[z]g[z] \Longrightarrow \lim_{\Delta z \to 0} \frac{F[z+\Delta z] - F[z]}{\Delta z} = \lim_{\Delta z \to 0} \frac{f[z+\Delta z]g[z+\Delta z] - f[z]g[z]}{\Delta z} \tag{1.104}$$

is obtained using $f[z+\Delta z] \approx f[z] + f'[z]\Delta z$ and $g[z+\Delta z] \approx g[z] + g'[z]\Delta z$ for differentiable functions and retaining only first-order terms,

$$\lim_{\Delta z \to 0} \frac{f[z+\Delta z]g[z+\Delta z] - f[z]g[z]}{\Delta z} = \lim_{\Delta z \to 0} \frac{f[z]g'[z]\Delta z + f'[z]g[z]\Delta z}{\Delta z} \tag{1.105}$$

such that

$$F[z] = f[z]g[z] \Longrightarrow F'[z] = f[z]g'[z] + f'[z]g[z] \tag{1.106}$$

By similar reasoning one can verify all standard differentiation rules, subject to obvious conditions on differentiability of the various parts. Perhaps the most important is the *chain rule*

$$F[z] = g[f[z]] \Longrightarrow F'[z] = \left(g'[w]f'[z]\right)_{w=f[z]} \tag{1.107}$$

provided that $f$ is differentiable at $z$ and that $g$ is differentiable at $w = f[z]$.

## 1.8 Properties of Analytic Functions

Suppose that $f[z] = u[x, y] + i v[x, y]$ is analytic in domain $D$ and suppose that the second partial derivatives of the component functions $u$ and $v$ are continuous in $D$ also. (We will soon prove that analytic functions are infinitely differentiable so that the component functions $u$ and $v$ must have continuous partial derivatives of all orders within $D$.) Differentiation of the CR equations then gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Longrightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.108}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Longrightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} \Longrightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{1.109}$$

Therefore, both the real and imaginary components of $f$ are *harmonic functions* that satisfy Laplace's equation. Furthermore, comparing the two-dimensional gradients

$$\vec{\nabla} u = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} = \hat{x}\frac{\partial v}{\partial y} - \hat{y}\frac{\partial v}{\partial x} = \hat{n} \times \vec{\nabla} v \tag{1.110}$$

$$\vec{\nabla} v = \hat{x}\frac{\partial v}{\partial x} + \hat{y}\frac{\partial v}{\partial y} = -\hat{x}\frac{\partial u}{\partial y} + \hat{y}\frac{\partial u}{\partial x} = -\hat{n} \times \vec{\nabla} u \tag{1.111}$$

$$\therefore \vec{\nabla} u \cdot \vec{\nabla} v = 0 \tag{1.112}$$

we find that lines of constant $u$ (*level curves*) are orthogonal to lines of constant $v$ anywhere that $f'[z] \neq 0$. (Here $\hat{n}$ represents the outward normal to the $xy$-plane.) If $u$ represents a potential function, then $v$ represents the corresponding stream function (lines of force), or vice versa.

Consider, for example, $f[z] = z^2$ with $u = x^2 - y^2$ and $v = 2xy$. If we interpret $v$ as an electrostatic potential, then $u$ represents lines of force. Figure 1.15 shows equipotentials as solid lines, positive in the first quadrant and alternating sign by quadrant, and lines of force as dashed lines. The arrows indicate the direction of the force, as prescribed by $-\vec{\nabla} v$. If electrodes were shaped with surfaces parallel to equipotentials, the interior field would act as an electrostatic quadrupole lens, focussing a beam of positively-charged particles along the 45° and 225° directions and defocussing along the 135° and 315° directions.

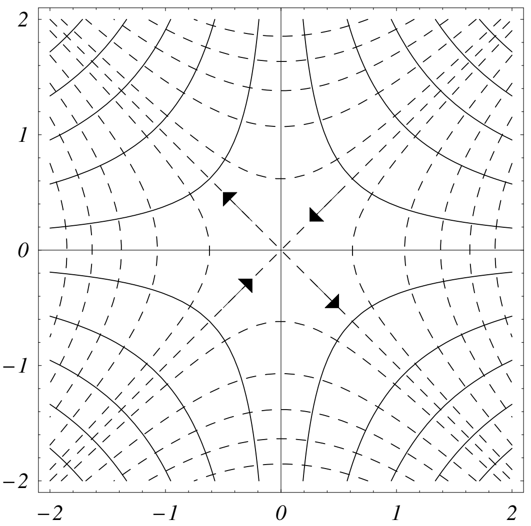


**Figure 1.15.** Level curves for $f[z] = z^2 = u + iv$ are shown as solid for $v$ and dashed for $u$. If the solid lines are interpreted as equipotentials, the dashed lines with directions given by $-\vec{\nabla}v$ represent lines of force.

Alternatively, if $v$ represents a magnetostatic potential, then $u$ would represent magnetic field lines. A beam of positively-charged particles moving into the page would be vertically focussed and horizontally defocussed by a magnetic quadrupole lens whose iron pole pieces have surfaces shaped by $v \propto xy$.

It is also easy to demonstrate that, although harmonic functions may have saddle points, they cannot have extrema in the finite plane. Hence, neither component of an analytic function may have an extremum within the domain of analyticity. Figure 1.16 illustrates the typical saddle shape for components of an analytic function. Furthermore, the average value of a harmonic function on a circle is equal to the value of that function of the center of the circle. Proofs of these hopefully familiar properties of Laplace's equation are left to the exercises.

Suppose that $Z_1$ is a curve in the $z$-plane represented by the parametric equations $z_1[t] = \{x_1[t], y_1[t]\}$ and that $f[z]$ is analytic in a domain containing $Z_1$, such that the image $W_1$ of that curve in the $w$-plane is represented by $w_1[t] = f[z_1[t]]$. The slopes of tangent lines at a point $z_0$ and its image $w_0$ are related by the chain rule, such that

$$w_1'[t] = f'[z]z_1'[t] \Longrightarrow \arg\left[w_1'[t]\right] = \arg\left[z_1'[t]\right] + \arg\left[f'[z_0]\right] \tag{1.113}$$

Thus, the mapping $f[z]$ rotates the tangent line through an angle $\arg[f'[z_0]]$. The tangent to a second curve which passes through the same point $z_0$ is rotated by the same amount,

$$w_2'[t] = f'[z]z_2'[t] \Longrightarrow \arg\left[w_2'[t]\right] = \arg\left[z_2'[t]\right] + \arg\left[f'[z_0]\right] \tag{1.114}$$

such that angle between the two curves

$$\arg[w_2'] - \arg[w_1'] = \arg[z_2'] - \arg[z_1'] \tag{1.115}$$

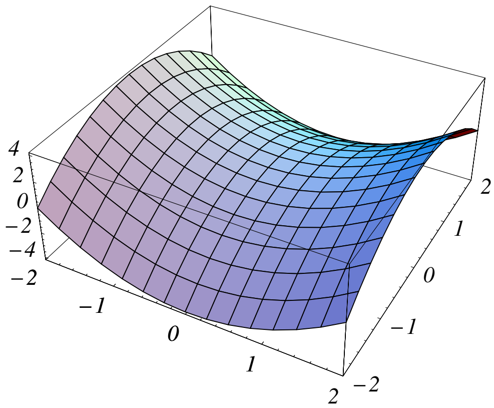


**Figure 1.16.** Typical saddle: $u = x^2 - y^2$.

is unchanged by the *conformal transformation* specified by an analytic function $f[z]$. Similarly, distances in the immediate vicinity of $z_0$ are scaled by the factor $|f'[z_0]|$, such that

$$|w - w_0| = |f'[z_0]||z - z_0| \tag{1.116}$$

Therefore, the image of a small triangle in the $z$-plane is a similar triangle in the $w$-plane that is generally rotated and scaled in size.

## 1.9 Cauchy–Goursat Theorem

### 1.9.1 Simply Connected Regions

We have seen that the components of analytic functions are harmonic and might be stimulated to pursue analogies with potential theory as far as possible. Remembering that the line integral about a closed path vanishes for a potential derived from a conservative force, we seek to evaluate

$$\oint_C f[z]\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx) \tag{1.117}$$

for an analytic function $f = u + iv$ of $z = x + iy$ where $u[x, y]$ and $v[x, y]$ are real. If we require $P[x, y]$ and $Q[x, y]$ to be differentiable within the simply connected region $R$ enclosed by the simple closed contour $C$, we can apply Stoke's theorem to prove

$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{1.118}$$

Let

$$\vec{V} = (P, Q, 0) \Longrightarrow \hat{n} \cdot \vec{\nabla} \times \vec{V} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \tag{1.119}$$

where $\hat{n}$ is normal to the $xy$-plane and use $d\vec{\lambda} = (dx, dy, 0)$ as the line element and $d\vec{\sigma} = \hat{n}\, dx\, dy$ as the area element to obtain

$$\oint_C d\vec{\lambda} \cdot \vec{V} = \int_R d\vec{\sigma} \cdot \vec{\nabla} \times \vec{V} \Longrightarrow \oint_C (P\, dx + Q\, dy) = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \tag{1.120}$$

Applying this result, known as Green's theorem, to the real and imaginary parts of the line integral separately, and using the CR conditions for analytic functions, we find

$$\oint_C (u\, dx - v\, dy) = \int_R \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = 0 \tag{1.121}$$

$$\oint_C (u\, dy + v\, dx) = \int_R \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy = 0 \tag{1.122}$$

and conclude that

$$f \text{ analytic for } z \text{ within } C \Longrightarrow \oint_C f[z]\, dz = 0 \tag{1.123}$$

This result was first obtained by Cauchy, but was later generalized by Goursat. The derivation above requires not only that $f'[z]$ exist throughout $R$, but also that it be continuous therein. The latter restriction can be removed.

**Theorem 5.** *Cauchy–Goursat theorem: If a function $f[z]$ is analytic at all points on and within a simple closed contour C, then $\oint_C f[z]\, dz = 0$.*

### 1.9.2 Proof

Consider the closed contour $C$ sketched in Fig. 1.17. Divide the enclosed region $R$ into a grid of squares and partial squares, whereby

$$\oint_C f[z]\, dz = \sum_{j=1}^{n} \oint_{C_j} f[z]\, dz \tag{1.124}$$

where the contributions made by shared interior boundaries cancel such that the net contour integral is the sum of the exterior borders of outer partial squares. For each of these cells, we construct the function

$$\delta[z, z_j] = \frac{f[z] - f[z_j]}{z - z_j} - f'[z_j] \tag{1.125}$$

where $z$ and $z_j$ are distinct points within or on $C_j$ and evaluate its largest modulus

$$\delta_j = \text{Max}\left[ \left| \frac{f[z] - f[z_j]}{z - z_j} - f'[z_j] \right| \right] \tag{1.126}$$

For any positive value of $\varepsilon$, a finite number of subdivisions is sufficient to ensure that all $\delta_j < \varepsilon$ because $f[z]$ is differentiable. Thus, we can now write

$$f[z] = f[z_j] + \left( f'[z_j] + \delta[z, z_j] \right)(z - z_j) \tag{1.127}$$

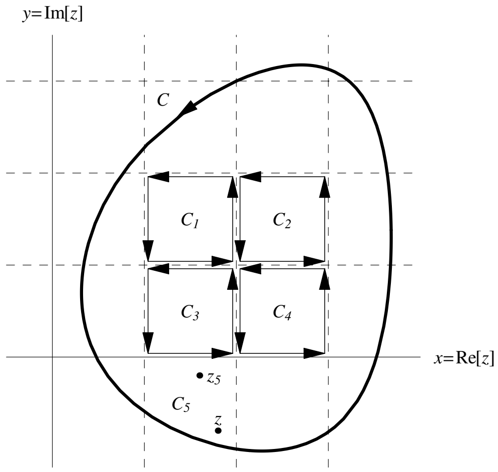


**Figure 1.17.** Proof of the Cauchy–Goursat theorem. Contours about four of the interior squares are labeled $C_{1-4}$. If $f[z]$ is continuous, contributions to the contour integral from shared sides cancel, leaving only the outer border $C$ that passes through partial squares. In the partial square, labeled $C_5$, we identify two distinct points labeled $z$ and $z_5$.

for any $z \subset C_j$, such that

$$\oint_{C_j} f[z]\, dz = f[z_j] \oint_{C_j} dz + f'[z_j] \oint_{C_j} (z - z_j)\, dz + \oint_{C_j} \delta[z, z_j](z - z_j)\, dz \tag{1.128}$$

The first two terms obviously vanish, leaving

$$\oint_C f[z]\, dz = \sum_{j=1}^{n} \oint_{C_j} \delta[z, z_j](z - z_j)\, dz \tag{1.129}$$

which can be bounded by

$$\left| \oint_C f[z]\, dz \right| \le \sum_{j=1}^{n} \left| \oint_{C_j} \delta[z, z_j](z - z_j)\, dz \right| \tag{1.130}$$

If $s_j$ is the length of the longest side of partial square $C_j$, then $|z - z_j| \le \sqrt{2} s_j$. Furthermore, $|\delta_j| < \varepsilon$, such that

$$\left| \oint_{C_j} \delta[z, z_j](z - z_j)\, dz \right| \le \sqrt{2} s_j \varepsilon (4 s_j + L_j) \tag{1.131}$$

where $L_j$ is the length of that part of $C_j$ that coincides with $C$. Because each factor is bounded and $\varepsilon \to 0$ may be taken arbitrarily small, we find that $|\oint_C f[z]\, dz|$ is also arbitrarily small and, hence, must vanish. Therefore, the Cauchy–Goursat theorem is established without assuming that $f'$ is continuous.

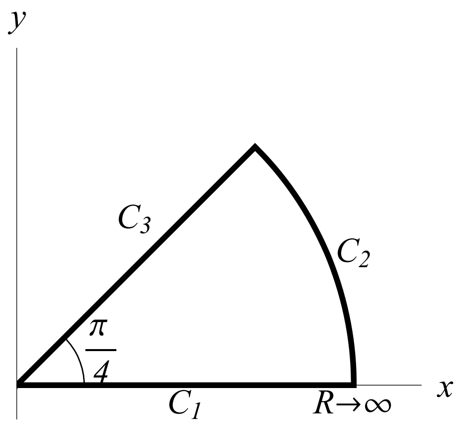


**Figure 1.18.** Wedge contour used for $\int_0^\infty \mathrm{Cos}[x^2]\, dx$.

### 1.9.3 Example

Contour integration of analytic functions provides powerful new methods for evaluation of otherwise intractable definite integrals. Although we will consider a wider variety later, for now consider the integral

$$\int_0^\infty \mathrm{Cos}[x^2]\, dx \tag{1.132}$$

which arises in the Fresnel theory of diffraction. It appears to be difficult to evaluate this integral using standard methods for real variables; nor is it obvious that this integral even converges. On the other hand, the Cauchy–Goursat theorem ensures that

$$I = \oint_C \mathrm{Exp}[iz^2]\, dz = 0 = I_1 + I_2 + I_3 \tag{1.133}$$

for a contour $C$ consisting of a wedge of opening angle $\theta = \frac{\pi}{4}$ closed by a circular arc at $R \to \infty$; this contour is shown in Fig. 1.18. Consider first the circular arc where

$$z = Re^{i\theta} \Longrightarrow e^{iz^2} = \mathrm{Exp}\left[iR^2 \mathrm{Cos}[2\theta]\right] \mathrm{Exp}\left[-R^2 \mathrm{Sin}[2\theta]\right] \tag{1.134}$$

Recognizing that $0 < \mathrm{Sin}[2\theta] < 1$ is positive on the arc, the integrand is damped by a factor of order $e^{-R^2}$ such that

$$R \to \infty \Longrightarrow I_2 = 0 \Longrightarrow I_1 = -I_3 \tag{1.135}$$

where

$$I_1 = \int_0^\infty \mathrm{Cos}[x^2]\, dx + i \int_0^\infty \mathrm{Sin}[x^2]\, dx \tag{1.136}$$

The return line is represented by

$$z = \frac{1+i}{\sqrt{2}} t \Longrightarrow dz = \frac{1+i}{\sqrt{2}}\, dt\; e^{iz^2} = e^{-t^2} \tag{1.137}$$

such that

$$I_3 = \frac{1+i}{\sqrt{2}} \int_{\infty}^{0} e^{-t^2}\, dt = -\frac{1+i}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \tag{1.138}$$

Therefore, equating real and imaginary parts, we find

$$\int_0^\infty \mathrm{Cos}[x^2]\, dx = \int_0^\infty \mathrm{Sin}[x^2]\, dx = \sqrt{\frac{\pi}{8}} \tag{1.139}$$

rather easily. By representing the integrand in terms of analytic functions and choosing a clever contour, one can perform a surprisingly diverse variety of integrals relatively painlessly. In this case we even obtain two results for the price of one. (What a deal!)

## 1.10 Cauchy Integral Formula

### 1.10.1 Integration Around Nonanalytic Regions

Suppose that the region $R = R_1 + R_2$ enclosed by the simple closed contour $C$ includes a localized region $R_2$ where the function $f$ is nonanalytic, but that $f$ is analytic everywhere else within $C$. The Cauchy–Goursat theorem can be applied to such a region by deforming the contour in a manner that encapsulates the problematic region. Figure 1.19 illustrates this technique. The colored region represents the nonanalytic region $R_2$ and the outer circle, when closed, represents the contour $C$ and is traversed in a positive, counterclockwise, sense. Note that $C$ need not actually be circular, but it is easier to draw that way. We imagine drawing line $A$ from $C$ to a point just outside the nonanalytic region. The contour $C_2$ goes around this region in a negative, clockwise sense, remaining within the analytic region $R_1$, ending close to its starting point. We then return along $B$ to the contour $C_1$. The common path $AB$ traversed in opposite directions between inner and outer contours is sometimes called a *contour wall* and serves to create a simply connected region $R_1$ for which the Cauchy–Goursat theorem requires

$$\int_{C_1} f[z]\, dz + \int_A f[z]\, dz + \int_B f[z]\, dz + \int_{C_2} f[z]\, dz = 0 \tag{1.140}$$

Recognizing that, for a continuous integrand, the contributions of $A$ and $B$ must become equal and opposite as the separation between those paths becomes infinitesimal, we find

$$\int_A f[z]\, dz + \int_B f[z]\, dz = 0 \Longrightarrow \oint_C f[z]\, dz = -\oint_{C_2} f[z]\, dz \tag{1.141}$$

Here the negative sign occurs because the inner contour is traversed in the opposite direction when reached by means of the contour wall. Therefore, the original contour can be *shrink-wrapped* about the nonanalytic region without changing the value of the contour integral.

If the path $C$ encloses several localized nonanalytic regions, we simply construct several contour walls. The net contour integral is then just the sum of the contributions

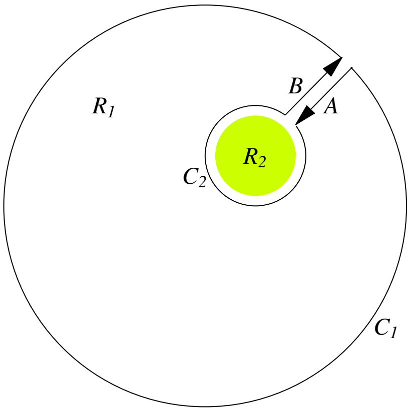


**Figure 1.19.** Construction of a contour wall and demonstration that a contour within an analytic region may be shrink-wrapped around and enclosed nonanalytic region.

from shrink-wrapped contours around each nonanalytic region. Take care with the signs though – if the original contour is traversed in a positive sense, the nonanalytic regions are enclosed in a negative sense by the continuous deformed contour that circumvents nonanalytic regions. However, recognizing that the entire contour integral vanishes and that the contour walls cancel, the net integral for a simple contour that encloses nonanalytic regions reduces to the sum of the contributions made by shrink-wrapped contours enclosing the nonanalytic regions in a positive sense. Therefore, if there are $N$ isolated nonanalytic regions within the simple closed contour $C$, we find

$$\oint_C f[z]\, dz = \sum_{k=1}^{N} \oint_{C_k} f[z]\, dz \tag{1.142}$$

where each simple closed contour $C_k$ encloses one of the nonanalytic regions and is traversed with the same sense as the original contour $C$.

We postpone consideration of extended nonanalytic regions to the next chapter, but in the next few sections consider the important special case of an isolated singularity within the contour.

### 1.10.2 Cauchy Integral Formula

Suppose that the contour $C$ lies within a region $R$ in which $f[z]$ is analytic, but that it surrounds another region $R'$ in which $f$ is not analytic. We demonstrated above that the contour can be deformed, such that $C \to C'$ where $C'$ is immediately outside $R'$, without changing the value of the contour integral

$$\oint_C f[z]\, dz = \oint_{C'} f[z]\, dz \tag{1.143}$$

Thus, a contour integral that encloses a single localized nonanalytic region can be shrink-wrapped about the border of that region. This result is particularly useful for the case of an isolated singularity for which the region of nonanalyticity consists of a single point $z_0$. Consider the integral

$$\oint_C ds \frac{f[s]}{s-z} \tag{1.144}$$

where $f$ is analytic throughout the region enclosed by $C$ while the integrand it singular at $z$. If $z$ is outside $C$ the integral vanishes because the integrand is analytic at all points within $C$. Alternatively, if $z$ lies within $C$, we can reduce $C$ to a small circle surrounding $z$, such that

$$s - z = re^{i\theta} \Longrightarrow ds = ire^{i\theta}\, d\theta \tag{1.145}$$

Thus, the integral can be approximated

$$\oint_C ds \frac{f[s]}{s-z} \approx f[z] \oint_C \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = 2\pi i f[z] \tag{1.146}$$

to arbitrary accuracy as $r \to 0$. Therefore, we obtain the *Cauchy integral formula*:

**Theorem 6.** *Cauchy integral formula: If a function $f[z]$ is analytic at all points on and within a simple closed contour C, then $f[z] = \frac{1}{2\pi i} \oint_C \frac{f[s]}{s-z}\, ds$ for any interior point z.*

This remarkably powerful theorem requires that the value of an analytic function at any interior point is uniquely determined by its values on any surrounding closed curve and is analogous to the two-dimensional form of Gauss' theorem. The behavior of an analytic function is severely constrained.

### 1.10.3 Example: Yukawa Field

Using elementary field theory, the virtual pion field surrounding a nucleon is represented in momentum space by

$$\tilde{\phi}[q] = \frac{\Lambda^2}{q^2 + \Lambda^2} \tag{1.147}$$

The spatial distribution is then obtained from the three-dimensional Fourier transform

$$\phi[r] = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \tilde{\phi}[q] = \frac{4\pi}{(2\pi)^3} \frac{\Lambda^2}{r} \int_0^\infty \frac{q \operatorname{Sin}[qr]}{q^2 + \Lambda^2}\, dq \tag{1.148}$$

where spherical symmetry and the multipole expansion of the plane wave have been used to reduce the integral to one dimension. (Alternatively, the angular integrals can be evaluated directly.) Recognizing that the integrand is even, we can write

$$\int_0^\infty \frac{q \operatorname{Sin}[qr]}{q^2 + \Lambda^2}\, dq = \frac{1}{2} \int_{-\infty}^\infty \frac{q \operatorname{Sin}[qr]}{q^2 + \Lambda^2}\, dq = \frac{1}{2i} \int_{-\infty}^\infty \frac{q \operatorname{Exp}[iqr]}{q^2 + \Lambda^2}\, dq \tag{1.149}$$

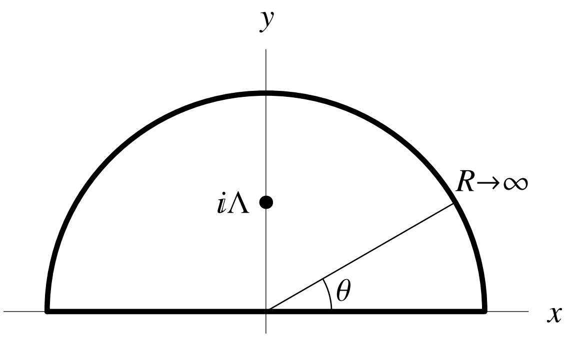


**Figure 1.20.** Great semicircle with enclosed pole at $z = i\Lambda$.

because the contribution from Cos[$qr$] vanishes by symmetry. Now consider the contour integral

$$I[\Lambda] = \oint_C \frac{g[z]}{z - i\Lambda}\, dz \tag{1.150}$$

where

$$g[z] = \frac{z e^{izr}}{(z + i\Lambda)} \tag{1.151}$$

is analytic in the upper half-plane. If we choose a contour $C$, shown in Fig. 1.20, consisting of the real axis and a semicircle in the upper half-plane with $R \to \infty$, affectionately called a *great semicircle*, this integral can be expressed as

$$I[\Lambda] = \int_{-\infty}^{\infty} \frac{q\,\mathrm{Exp}[iqr]}{q^2 + \Lambda^2}\, dq + iR^2 \int_0^{\pi} \frac{\mathrm{Exp}\left[irRe^{i\theta}\right]}{R^2 e^{2i\theta} + \Lambda^2}\, d\theta \tag{1.152}$$

Using

$$\mathrm{Exp}\left[irRe^{i\theta}\right] = \mathrm{Exp}[irR\,\mathrm{Cos}[\theta]]\,\mathrm{Exp}[-rR\,\mathrm{Sin}[\theta]] \tag{1.153}$$

and recognizing that $\mathrm{Sin}[\theta] > 0$ in the upper half-plane, we realize that the contribution of the circular arc decreases exponentially with $R$ and vanishes in the limit $R \to \infty$. Therefore, with the aid of the Cauchy integral formula

$$I[\Lambda] = 2\pi i g[i\Lambda] = i\pi e^{-\Lambda r} \tag{1.154}$$

we obtain the Yukawa field

$$\phi[r] = \frac{\Lambda^2}{4\pi} \frac{e^{-\Lambda r}}{r} \tag{1.155}$$

that is central to the meson exchange model of the nucleon–nucleon interaction that binds atomic nuclei together.

### 1.10.4 Derivatives of Analytic Functions

Let

$$f[z] = \frac{1}{2\pi i}\oint_C dw \frac{f[w]}{w-z} \tag{1.156}$$

represent a function that is analytic in the domain $D$ containing the simple closed contour $C$. First we demonstrate that differentiation can be performed under the integral sign. Using

$$\begin{aligned} f[z+\Delta z] - f[z] &= \frac{1}{2\pi i}\oint_C dw \left( \frac{f[w]}{w-z-\Delta z} - \frac{f[w]}{w-z} \right) \\ &= \frac{\Delta z}{2\pi i}\oint_C dw \frac{f[w]}{(w-z)(w-z-\Delta z)} \end{aligned} \tag{1.157}$$

we recognize that the left-hand side vanishes in the limit $\Delta z \to 0$ because $f[z]$ is continuous and must demonstrate that the right-hand side shares this property. Recognizing that the integrand is analytic everywhere within $C$ except at $z$, we may reduce the contour to a small circle of radius $r$ around $z$. Let $M = \max[|f[w]|]$ on the reduced contour, and use the triangle inequalities to evaluate the maximum modulus of the integrand, such that

$$\begin{aligned} \left| \Delta z \oint_C dw \frac{f[w]}{(w-z)(w-z-\Delta z)} \right| &\leq |\Delta z| \oint_C dw \frac{M}{|(w-z)(w-z-\Delta z)|} \\ &\leq 2\pi r \frac{M|\Delta z|}{r(r-|\Delta z|)} \end{aligned} \tag{1.158}$$

vanishes in the limit $\Delta z \to 0$ for finite $r$. Thus, we find that

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f[z+\Delta z] - f[z]}{\Delta z} &= \frac{1}{2\pi i}\oint_C dw \frac{f[w]}{(w-z)^2} \\ &\Longrightarrow f'[z] = \frac{1}{2\pi i}\oint_C dw f[w] \frac{d}{dz}(w-z)^{-1} \end{aligned} \tag{1.159}$$

is also analytic within $D$. Repeating this process, we obtain

$$f^{(n)}[z] = \frac{d^n f[z]}{dz^n} = \frac{n!}{2\pi i}\oint_C dw \frac{f[w]}{(w-z)^{n+1}} \tag{1.160}$$

by induction. Therefore, we have demonstrated by construction the remarkably powerful theorem that analytic functions have derivatives of all orders. This also requires all partial derivatives of its component functions to be continuous in $D$. This theorem will soon be used to derive series representations of analytic functions.

**Theorem 7.** *If a function $f$ is analytic at a point, then derivatives of all orders exist and are analytic at that point.*

### 1.10.5 Morera's Theorem

The converse of Cauchy–Goursat theorem is known as *Morera's theorem*:

**Theorem 8.** *Morera's theorem: If a function $f[z]$ is continuous in a simply connected region R and $\oint_C f[z]\,dz = 0$ for every simple closed contour C within R, then $f[z]$ is analytic throughout R.*

If every closed path integral vanishes, the path integral between two points in the domain of analyticity $D$ depends only upon the end points and is independent of the path, provided that the path lies entirely within $D$. Hence, we define the function $F[z]$ by means of the definite integral

$$F[z_2] - F[z_1] = \int_{z_1}^{z_2} f[z]\,dz \tag{1.161}$$

Clearly,

$$\int_{z_1}^{z_2} \big(f[z] - f[z_1]\big)\,dz = F[z_2] - F[z_1] - (z_2 - z_1)f[z_1] \tag{1.162}$$

such that the limit as $z_2 \to z_1$

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} \big(f[z] - f[z_1]\big)\,dz}{z_2 - z_1} = \lim_{z_2 \to z_1} \frac{F[z_2] - F[z_1]}{z_2 - z_1} - f[z_1] = F'[z_1] - f[z_1] \tag{1.163}$$

compares $F'[z_1]$ with $f[z_1]$. However, the integral vanishes in the limit of vanishing range of integration because $f[z]$ is continuous in $D$, such that

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} \big(f[z] - f[z_1]\big)\,dz}{z_2 - z_1} = 0 \Longrightarrow F'[z_1] = f[z_1] \tag{1.164}$$

Thus, $F[z]$ is analytic in $D$ with $F'[z] = f[z]$. Therefore, because the derivative of an analytic function is also analytic, we conclude that $f[z]$ must also be analytic, proving Morera's theorem.

Morera's theorem is sometimes useful for proving general properties for analytic functions of various types, but is rarely of practical value to more detailed calculations.

## 1.11 Complex Sequences and Series

### 1.11.1 Convergence Tests

An infinite sequence of complex numbers $\{z_n, n = 1, 2, \ldots\}$ can be represented by combining two sequences of real numbers $\{x_n, n = 1, 2, \ldots\}$ and $\{y_n, n = 1, 2, \ldots\}$ such that $z_n = x_n + iy_n$. The sequence $z_n$ converges to $z$ if for any small positive $\epsilon$ there exists an integer $N$ such that $|z_n - z| < \epsilon$ for $n > N$. Convergence of a complex series $z_n$ to $z = x + iy$ then requires convergence of both $x_n$ to $x$ and $y_n$ to $y$. Many of the properties of real sequences can be adapted to complex sequences with only minor and obvious changes. Therefore, we state without proof the *Cauchy convergence principle*:

**Theorem 9.** *The sequence $\{z_n\}$ converges if and only if for every small positive $\epsilon$ there exists an integer $N_\epsilon$ such that $|z_m - z_n| < \epsilon$ for any $m, n > N_\epsilon$.*

If $\{z_n\}$ and $\{w_n\}$ are two convergent sequences with limits $z$ and $w$, then $\{az_n + bw_n\}$ and $\{z_n w_n\}$ are also convergent sequences with limits $az + bw$ and $zw$.

An infinite series of complex numbers $z_k$ converges if the sequence of partial sums

$$S_n = \sum_{k=1}^{n} z_k \tag{1.165}$$

converges to $S$, such that

$$\lim_{n\to\infty} S_n = S \Longrightarrow S = \sum_{k=1}^{\infty} z_k \tag{1.166}$$

If the sequence of partial sums does not converge, the corresponding series diverges. A series is *absolutely convergent* if the series of moduli

$$\sum_{k=1}^{n} |z_k| \tag{1.167}$$

converges. An absolutely convergent series converges, but a convergent series need not converge absolutely. A convergent series that is not absolutely convergent is described as *conditionally convergent*. For example, the alternating harmonic series $\sum_{k=1}^{\infty}(-)^k/k$ converges conditionally but not absolutely because $\sum_{k=1}^{\infty} k^{-1}$ diverges. Term-by-term addition of convergent series yields another convergent series, but convergence of a series formed by termwise multiplication requires absolute convergence of the individual series.

If $\{z_k\}$ does not converge to zero, the corresponding series diverges because the sequence of partial sums will not satisfy the Cauchy convergence condition. However, convergence of the sequence of terms to zero does not ensure convergence of the series. The most general analysis of a series separates its terms into real and imaginary parts and then applies one of the many tests developed for series of real numbers to the real and imaginary subseries separately; the complex series then converges if both its real and imaginary subseries converge. However, it is usually simpler and often sufficient to test for absolute convergence instead. The following convergence tests familiar for real series can be generalized to complex series.

***Comparison test:*** If $0 \le |z_k| \le a_k$ for sufficiently large $k$ and $\sum_k a_k$ converges, then $\sum_k z_k$ converges absolutely.

***Ratio test:*** If $|z_{k+1}/z_k| \le r$ for all $k > N$, then $\sum_k z_k$ converges absolutely if $r < 1$. Alternatively, if $r = \lim_{k\to\infty} |z_{k+1}/z_k|$ the series converges absolutely if $r < 1$ but diverges if $r > 1$. This test is inconclusive if $r = 1$.

***Root test:*** If $|z_k|^{1/k} \le r$ for all $k > N$, then $\sum_k z_k$ converges absolutely if $r < 1$. Alternatively, if $r = \lim_{k\to\infty} |z_k|^{1/k}$ the series converges absolutely if $r < 1$ but diverges if $r > 1$. This test is also inconclusive if $r = 1$.

***Integral test:*** Suppose that $f[k] = |z_k|$ where $f[x]$ is defined for $x \geq n \geq 1$. The series then converges absolutely if the integral $\int_n^\infty f[x]\, dx$ converges.

Note that the ratio test is indeterminate when $\lim_{k\to\infty} |z_{k+1}/z_k| = 1$. For example, the harmonic series $z_k = k^{-1}$ diverges while the alternating harmonic series converges. A "sharpened" version, established in the exercises, shows that a series converges absolutely if the ratio of successive terms takes the form

$$\left|\frac{a_{n+1}}{a_n}\right| \simeq 1 - \frac{s}{n} \tag{1.168}$$

for large $n$ with $s > 1$.

Often the terms of a series will themselves be functions of a complex variable, $z$, such that

$$f_n[z] = \sum_{k=1}^{n} g_k[z] \tag{1.169}$$

represents a sequence $\{f_n[z]\}$ of partial sums. If such a sequence converges for all $z$ in a region $R$, such that

$$z \in R \Longrightarrow f[z] = \lim_{n\to\infty} f_n[z] = \lim_{n\to\infty} \sum_{k=1}^{n} g_k[z] \tag{1.170}$$

then $f_n[z]$ is described as a series *representation* of the function $f[z]$ valid within the *convergence region* $R$. Often the convergence region takes the form of a disk, $|z - z_0| \leq R$, with center $z_0$ and *radius of convergence* $R$. If a series converges for all $z$ within $|z - z_0| < R$ but diverges for some points on the circle $|z - z_0| = R$, one still reports a radius of convergence $R$. The problem then is to determine the radius of convergence.

## Example

What is the radius of convergence for a geometric series, $\sum_{k=0}^{\infty} z^k$, extended to the complex plane? According to the ratio test,

$$\left|\frac{z_{k+1}}{z_k}\right| = \left|\frac{z^{k+1}}{z^k}\right| = |z| \tag{1.171}$$

this series converges absolutely for any $|z| < 1$ and diverges for $|z| > 1$. Thus, the radius of convergence is 1. Notice that even though the ratio test is inconclusive for $|z| = 1$, this series clearly diverges on the unit circle because the terms do not approach zero. Alternatively, by the ratio test

$$\lim_{k\to\infty} |z_k|^{1/k} = |z| \tag{1.172}$$

one finds convergence for $|z| > 1$ and divergence for $|z| \geq 1$. Furthermore, one can demonstrate that

$$|z| < 1 \Longrightarrow \lim_{n\to\infty} \sum_{k=0}^{n} z^k = \frac{1}{1-z} \tag{1.173}$$

within the radius of convergence. Let

$$f_n[z] = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \tag{1.174}$$

represent a sequence of complex numbers, where the last step is verified by direct multiplication

$$\begin{aligned}(1 - z)(1 + z + z^2 + \ldots + z^n) &= (1 + z + z^2 + \ldots + z^n) - (z + z^2 + \ldots + z^{n+1}) \\ &= 1 - z^{n+1}\end{aligned} \tag{1.175}$$

Then, separating the constant term (for fixed $z$) from the variable part of the sequence

$$f_n[z] = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z} \tag{1.176}$$

and recognizing

$$|z| < 1 \Longrightarrow \lim_{n\to\infty} \frac{z^{n+1}}{1 - z} = 0 \tag{1.177}$$

one finds that

$$|z| < 1 \Longrightarrow \lim_{n\to\infty} f_n[z] = \frac{1}{1 - z} \tag{1.178}$$

Therefore, the geometric series

$$|z| < 1 \Longrightarrow \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \tag{1.179}$$

converges to a simple analytic function within the unit circle, thereby extending a familiar result from the real axis to the complex plane.

### 1.11.2 Uniform Convergence

A sequence of functions $\{f_n[z]\}$ is said to *converge uniformly* to the function $f[z]$ in a region $R$ if there exists a fixed positive integer $N_\epsilon$ such that $|f_n[z] - f[z]| < \epsilon$ for any $z$ within $R$ when $n > N_\epsilon$. Consequently, a uniformly convergent series $f_n[z] = \sum_{k=1}^{n} g_k[z]$ provides an approximation to $f[z]$ within $R$ with controllable accuracy – there exists a finite number of terms, even if large, that guarantees a specified degree of accuracy anywhere within the region of uniform convergence. The region of uniform convergence is always a subset of the region of convergence. For example, although the geometric series $\sum_{k=0}^{\infty} z^k$ converges uniformly to $(1 - z)^{-1}$ within any disk $|z| \leq R < 1$ with less than unit radius and is convergent within $|z| < 1$, one cannot properly claim uniform convergence throughout the open region $|z| < 1$ because the convergence becomes so slow near the circle of convergence that there will always be points within that region that require more than

$N$ terms to achieve the desired accuracy no matter how large $N$ is chosen. Convergence at $z$ without uniform convergence within the region of interest is described as *pointwise*.

The most common test for uniform convergence is offered by the *Weierstrass M-test*: The series $\sum_k f_k[z]$ is uniformly convergent in region $R$ if there exists a series of positive constants $M_k$ such that $|f_k[z]| \leq M_k$ for all $z$ in $R$ and $\sum_k M_k$ converges. The proof follows directly from the comparison test. (For what it's worth, $M$ stands for *majorant*.)

The follow theorems for manipulation of uniformly convergent series can be established by straightforward generalization of the corresponding results for real functions.

***Continuity theorem*:** a uniformly convergent series of continuous functions is continuous.

***Combination theorem*:** the sum or product of two uniformly convergent series is uniformly convergent within the overlap of their convergence regions.

***Integrability theorem*:** the integral of a uniformly convergent series of continuous functions is equal to the sum of the integrals of each term.

***Differentiability theorem*:** the derivative of a uniformly convergent series of continuous functions with continuous derivatives is uniformly convergent and is equal to the sum of the derivatives of each term.

Furthermore, by combining these results one can obtain the more general *Weierstrass theorem* establishing uniformly convergent series as analytic functions within their convergence regions. Thus, the property of uniform convergence is important because it makes available all theorems in the theory of analytic functions.

**Theorem 10.** *Weierstrass theorem: If the terms of a series $\sum_k g_k[z]$ are analytic throughout a simply-connected region R and the series converges uniformly throughout R, then its sum is an analytic function within R and the series may be integrated or differentiated termwise any number of times.*

## 1.12 Derivatives and Taylor Series for Analytic Functions

### 1.12.1 Taylor Series

It is now a simple matter to demonstrate the existence of power-series expansions for analytic functions. Suppose that $f$ is analytic within a disk $|z - z_0| \leq R$ centered upon $z_0$ and assume that

$$f[z] = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.180}$$

Consider the integral

$$I_k = \frac{1}{2\pi i} \oint dz \frac{1}{(z - z_0)^k} \tag{1.181}$$

evaluated on the circle $|z - z_0| = R$. Using $dz = iRe^{i\theta}\,d\theta$, we find that

$$I_k = \frac{R^{1-k}}{2\pi}\int_0^{2\pi} e^{i(1-k)\theta}\,d\theta = \delta_{k,1} \tag{1.182}$$

vanishes unless $k = 1$. Therefore, the coefficients of the power series can be evaluated according to

$$a_n = \frac{1}{2\pi i}\oint dz \frac{f[z]}{(z-z_0)^{n+1}} = \frac{f^{(n)}[z_0]}{n!} \tag{1.183}$$

Although we performed this calculation using circular contours, the same results would be obtained for arbitrary simple contours within the analytic region because the singularities in the integrands are confined to a single point, which can be excised. A power series centered upon the origin is sometimes called a *Maclaurin series* while a more general power series about arbitrary $z_0$ is called a *Taylor series*.

**Theorem 11.** *Taylor series: If a function $f$ is analytic within a disk $|z - z_0| \leq R$, then the power series $f[z] = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with $a_n = \frac{f^{(n)}[z_0]}{n!}$ converges to $f[z]$ at all points within the disk. Conversely, if a power series converges for $|z - z_0| \leq R$, it represents an analytic function within that disk.*

It is instructive to demonstrate convergence of the power series directly. Expanding

$$(s-z)^{-1} = (s-z_0)^{-1}\left(1 - \frac{z-z_0}{s-z_0}\right)^{-1} = (s-z_0)^{-1}\left(A_n + \sum_{k=0}^{n}\left(\frac{z-z_0}{s-z_0}\right)^k\right) \tag{1.184}$$

where

$$A_n = \frac{(z-z_0)^{n+1}}{(s-z_0)^n (s-z)} \tag{1.185}$$

the Cauchy integral formula becomes

$$f[z] = \frac{1}{2\pi i}\oint_C ds \frac{f[s]}{s-z} = \sum_{k=0}^{n} a_k (z-z_0)^k + R_n (z-z_0)^{n+1} \tag{1.186}$$

where

$$a_k = \frac{1}{2\pi i}\oint ds \frac{f[s]}{(s-z_0)^{k+1}} = \frac{f^{(k)}[z_0]}{k!} \tag{1.187}$$

as before and where the remainder takes the form

$$R_n = \frac{1}{2\pi i}\oint_C ds \frac{f[s]}{(s-z)}\frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}} \tag{1.188}$$

Identifying

$$|z - z_0| = \rho, \quad M = \max[|f[s]|], \quad \delta = \min[|s-z|] \tag{1.189}$$

and choosing a circular contour with

$$|s - z_0| = r > \rho \tag{1.190}$$

we find

$$|R_n| \leq \left(\frac{\rho}{r}\right)^{n+1} \frac{r}{\delta} M \Longrightarrow \lim_{n\to\infty} R_n = 0 \tag{1.191}$$

Thus, this power series converges throughout the region of analyticity that surrounds $z_0$. Therefore, *the radius of convergence is the distance to the nearest singularity in the complex plane*. With more careful analysis, one may find that a Taylor series converges at some points on the circle of convergence also.

The Taylor series for $f[z]$ about a point $z_0 = (x_0, 0)$ on the real axis has the same form as the expansion of $f[x]$ interpreted as a function of the real variable $x$. More importantly, this extension of the Taylor theorem to the complex plane often provides the simplest method for evaluating the radius of convergence of a power series. Consider the hyperbolic tangent

$$\mathrm{Tanh}[z] = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 - \frac{17}{315}z^7 + \frac{62}{2835}z^9 - \frac{1382}{155\,925}z^{11} + \frac{21\,844}{6\,081\,075}z^{13} + \cdots \tag{1.192}$$

It is difficult to evaluate the general term and to deduce the radius of convergence from the real-valued series, but from the function of a complex variable we know immediately that the radius of convergence is $\pi/2$ because the nearest roots of Cosh$[z]$ are found at $z = \pm i\pi/2$.

Sometimes it is necessary to determine the radius of convergence directly from the terms of the power series. Then one finds

$$R = \left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} \tag{1.193}$$

using the ratio test, or

$$R = \lim_{n\to\infty} |a_n|^{-1/n} \tag{1.194}$$

using the root test. For example, the Maclaurin series for Log$[1 + z]$ takes the form

$$\mathrm{Log}[1 + z] = \sum_{n=1}^{\infty} (-)^{n+1} \frac{z^n}{n} \tag{1.195}$$

and one obtains a convergence radius $R = 1$ using the ratio test. In this case the convergence radius is limited by the branch point at $z = -1$. Notice that at $z = 1$ this power series reduces to the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} = \mathrm{Log}[2] \tag{1.196}$$

and thus converges for at least one point on the convergence circle, while at $z = -1$ the resulting harmonic series diverges. Applying the root test instead suggests a limit

$$\lim_{n\to\infty} n^{1/n} = 1 \tag{1.197}$$

that might not be obvious otherwise.

### 1.12.2 Cauchy Inequality

Let

$$M[r] = \max_{|z-z_0|=r}|f[z]| \tag{1.198}$$

represent the maximum modulus of an analytic function on a circle of radius $r$ surrounding $z_0$. We then find that

$$|a_n| \leq \frac{1}{2\pi}\oint dz\frac{M}{r^{n+1}} = Mr^{-n} \Longrightarrow |a_n|r^n \leq M[r] \tag{1.199}$$

constrains the coefficients of the Taylor series.

**Theorem 12.** *Cauchy inequality: If a function* $f[z] = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *is analytic and bounded in D and* $|f[z]| \leq M$ *on a circle* $|z-z_0| = r$*, then* $|a_n|r^n \leq M$.

### 1.12.3 Liouville's Theorem

**Theorem 13.** *Liouville's theorem: If a function* $f[z]$ *is analytic and bounded everywhere in the complex plane, then* $f[z]$ *is constant.*

According to the Cauchy inequality, if $|f[z]| < M$ for $|z| < R$, then $|a_n|R^n < M$. If this inequality applies in the limit $R \to \infty$, then we must require $a_n \to 0$ for $n > 0$. Therefore, if $f$ is not constant, it must have a singularity somewhere. The behavior of functions of a complex variable is largely determined by the nature and locations of their singularities.

### 1.12.4 Fundamental Theorem of Algebra

**Theorem 14.** *Fundamental theorem of algebra: Any polynomial* $P_n[z] = \sum_{k=0}^{n} a_n z^n$ *of order* $n \geq 1$ *must have at least one zero* $z_0 \ni P_n[z_0] = 0$ *in the finite complex plane.*

Although it is difficult to prove the fundamental theorem of algebra using purely algebraic means, it is an almost trivial consequence of Liouville's theorem. If $P_n[z]$ has no zeros, then the function $f[z] = 1/P_n[z]$ would be analytic throughout the entire complex plane. Recognizing that

$$|z| = R \to \infty \Longrightarrow |P_n[z]| \to |a_n|R^n \Longrightarrow f[z] \to \frac{1}{|a_n|R^n} \tag{1.200}$$

it is clear that $f[z]$ is bounded. Thus, Liouville's theorem requires $f$ to be constant, but $f[z]$ vanishes in the limit $R \to \infty$, which contradicts the absence of zeros in $P_n$. Therefore, $P_n$ must have at least one zero. Factoring out this root, we write $P_n[z] = (z-z_1)P_{n-1}[z]$ and apply the theorem to $P_{n-1}$, concluding that it must also have a root if $n > 2$. Repeating this process, we determine that a polynomial of degree $n$ must have $n$ roots, although some might be repeated.

### 1.12.5 Zeros of Analytic Functions

If a function $f[z]$ is analytic at $z_0$ there exists a disk $|z - z_0| < R$ wherein the Taylor series converges, such that

$$|z - z_0| < R \Longrightarrow f[z] = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.201}$$

Suppose that $z_0$ was chosen to be a root of $f[z_0] == 0$, such that $a_0 = 0$. If $a_1 \neq 0$, we describe $z_0$ as a simple zero of $f$, but if all $a_n = 0$ with $n < m$ while $a_m \neq 0$, we describe $z_0$ as a zero of order $m$. It is then useful to express the Taylor series in the form

$$f[z] = (z - z_0)^m \phi[z] \tag{1.202}$$

where the auxiliary function

$$\phi[z] = \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n \tag{1.203}$$

employing the coefficients $a_n$ with $n \geq m$ has the nonzero value $\phi[z_0] = a_m$ at $z_0$. Clearly $\phi$ is continuous at $z_0$ and is analytic within the radius of convergence. Therefore, for any small positive number $\varepsilon$ there exists a corresponding radius $\delta$ such that

$$|\phi[z] - a_m| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta \tag{1.204}$$

Suppose there were another point $z_1$ in a neighborhood of $z_0$ where $\phi[z_1] = 0$, such that

$$\phi[z_1] = 0 \Longrightarrow |a_m| < \varepsilon \quad \text{whenever} \quad |z_1 - z_0| < \delta \tag{1.205}$$

can only be satisfied if $a_m = 0$, contrary to our assumption that $z_0$ is a zero of order $m$. Therefore, we conclude that if $f$ is analytic and does not vanish identically, there must exist a neighborhood around any root in which no other root is found; in other words, the roots of analytic functions are isolated.

**Theorem 15.** *Suppose that a function $f[z]$ is analytic at $z_0$ and that $f[z_0] = 0$. Then there must exist a neighborhood of $z_0$ containing no other zeros of $f$ unless $f$ vanishes identically.*

## 1.13 Laurent Series

### 1.13.1 Derivation

A more general expansion which is useful in an analytic region that surrounds a nonanalytic region is provided by the Laurent series.

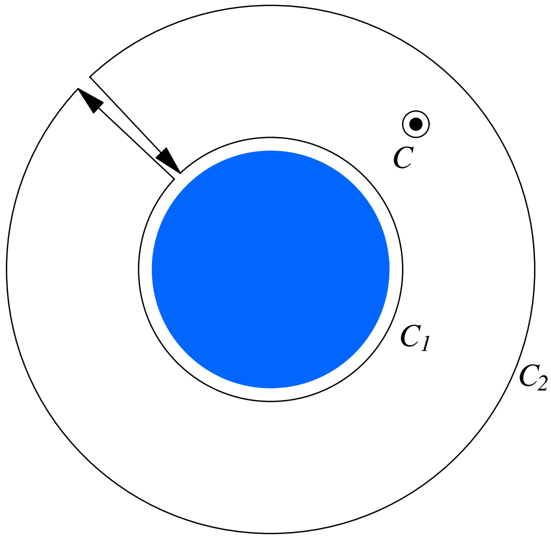


**Figure 1.21.** To develop the Laurent series, a small contour $C$ within an analytic region is stretched toward the limits of the region of analyticity, indicated by $C_1$ and $C_2$, with the aid of contour wall.

**Theorem 16.** *Laurent series: If $f[z]$ is analytic throughout the region $R_1 < |z - z_0| < R_2$, it can be represented by an expansion*

$$f[z] = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1.206}$$

*with coefficients*

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f[z]\, dz}{(z - z_0)^{n+1}} \tag{1.207}$$

*computed using any simple counterclockwise contour $C$ within the analytic region.*

If $R_1 \to 0$ and coefficients with $n < 0$ vanish, then the Laurent series reduces to the Taylor series.

Suppose that $C$ is a small contour surrounding an interior point $z$, such that

$$f[z] = \frac{1}{2\pi i} \oint_C ds \frac{f[s]}{s - z} \tag{1.208}$$

according to the Cauchy integral formula. As shown in Fig. 1.21, we can stretch $C$ to the limits of the annulus without changing the integral because the integrand is analytic throughout that region. Recognizing that the opposing segments of the contour wall cancel, we obtain

$$f[z] = \frac{1}{2\pi i} \oint_{R_2} ds \frac{f[s]}{s - z} - \frac{1}{2\pi i} \oint_{R_1} ds \frac{f[s]}{s - z} \tag{1.209}$$

where $R_1$ and $R_2$ denote counterclockwise circles at the inner and outer borders of the analytic annulus. For the outer integral we employ the expansion

$$(s-z)^{-1} = (s-z_0)^{-1}\left(1-\frac{z-z_0}{s-z_0}\right)^{-1} = \left(s-z_0\right)^{-1}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^n \tag{1.210}$$

while for the inner integral

$$(s-z)^{-1} = -(z-z_0)^{-1}\left(1-\frac{s-z_0}{z-z_0}\right)^{-1} = -\left(z-z_0\right)^{-1}\sum_{n=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^n \tag{1.211}$$

such that

$$2\pi i f[z] = \sum_{n=0}^{\infty}(z-z_0)^n \oint_{R_2} ds \frac{f[s]}{(s-z_0)^{1+n}} + \sum_{n=0}^{\infty}(z-z_0)^{-1-n}\oint_{R_1} ds f[s](s-z_0)^n \tag{1.212}$$

Both integrands are analytic throughout the annulus and are independent of $z$. Hence, these integrals can be evaluated using any simple closed path within the analytic region. Therefore, we may combine the two terms into a single expression

$$f[z] = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \tag{1.213}$$

$$a_n = \frac{1}{2\pi i}\oint_C dz \frac{f[z]}{(z-z_0)^{n+1}} \tag{1.214}$$

representing the Laurent expansion. One can also show that the Laurent expansion about a specific $z_0$ is unique within its analytic annulus.

### 1.13.2 Example

The function

$$f[z] = \frac{1}{z^2(1-z)} \tag{1.215}$$

has singular points at $z = 0, 1$. Suppose that we evaluate the Laurent coefficients using contour integration

$$a_n = \frac{1}{2\pi i}\oint_C ds \frac{f[s]}{s^{n+1}} = \frac{R^{-n-2}}{2\pi}\int_0^{2\pi} \frac{e^{-i\theta(n+2)}}{1-Re^{i\theta}}\, d\theta \tag{1.216}$$

on a circle with $s = Re^{i\theta}$ and $ds = is\, d\theta$. For $R < 1$ we can expand the integrand to obtain

$$R < 1 \Longrightarrow a_n = \frac{R^{-n-2}}{2\pi}\sum_{k=0}^{\infty} R^k \int_0^{2\pi} \mathrm{Exp}[i(k-n-2)\theta]\, d\theta \tag{1.217}$$

Nonvanishing coefficients then require $k = n+2$ and $k \geq 0 \Longrightarrow n \geq -2$, such that

$$0 < |z| < 1 \Longrightarrow f[z] = \sum_{n=0}^{\infty} z^{n-2} \tag{1.218}$$

Alternatively, for $R > 1$ we use $(1-z)^{-1} = -z^{-1}(1-z^{-1})^{-1}$ to obtain

$$R > 1 \implies a_n = -\frac{R^{-n-3}}{2\pi} \sum_{k=0}^{\infty} R^{-k} \int_0^{2\pi} \text{Exp}[-i(k+n+3)\theta]\, d\theta \tag{1.219}$$

for which nonvanishing coefficients require $k = -n-3$ and $k \geq 0 \implies n \leq -3$, such that

$$|z| > 1 \implies f[z] = -\sum_{n=0}^{\infty} z^{-n-3} \tag{1.220}$$

Although the Laurent theorem provides an explicit formula for the coefficients, evaluation of the contour integrals is often difficult and one seeks simpler alternative methods. In this case we can use the partial fractions

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z} \tag{1.221}$$

and

$$|z| < 1 \implies \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{1.222}$$

$$|z| > 1 \implies \frac{1}{1-z} = -\frac{1}{z(1-z^{-1})} = -\sum_{n=1}^{\infty} z^{-n} \tag{1.223}$$

to obtain the same results without integration. In other cases we may be able to convert a known Taylor series into a Laurent series. For example,

$$\text{Log}[1+z] = \sum_{n=1}^{\infty} (-)^{n+1} \frac{z^n}{n} \quad \text{for} \quad |z| < 1 \tag{1.224}$$

$$\implies \text{Log}\left[1+z^{-1}\right] = \sum_{n=1}^{\infty} (-)^{n+1} \frac{z^{-n}}{n} \quad \text{for} \quad |z| > 1 \tag{1.225}$$

where the latter is valid in the largest annulus that excludes the branch cut $-1 \leq x \leq 1$ on the real axis.

### 1.13.3 Classification of Singularities

Suppose that $f[z]$ is singular at $z_0$ but analytic at all other points in a neighborhood of $z_0$; $f$ is then said to have an *isolated singularity* at $z_0$. A function that is analytic throughout the finite complex plane except for isolated singularities is described as *meromorphic*. Meromorphic functions include entire functions, such as Exp, that have no singularities in the finite plane and rational functions that have a finite number of poles. Functions, such as Log, that require branch cuts are not meromorphic.

The Laurent expansion about an isolated singularity takes the form

$$f[z] = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1.226}$$

If $a_m \neq 0$ for some $m < 0$ while all $a_{n<m} = 0$, then $z_0$ is classified as a *pole of order* $-m$ and the coefficient $a_{-1}$ is called the *residue* of the pole. A *simple pole* has $m = -1$. If the function appears to have a singularity at $z_0$ but all $a_m$ vanish for $m < 0$, $z_0$ is described as a *removable singularity* because the function can be made analytic simply by assigning a suitable value to $f[z_0]$. For example, $z = 0$ is a removable singularity of

$$f[z] = \frac{\mathrm{Sin}[z]}{z} = \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)!} z^{2n} \tag{1.227}$$

because with the assignment $f[0] = 1$ the function is continuous and its Laurent series reduces to a simple Taylor series.

If the Laurent expansion has nonvanishing coefficients for arbitrarily large negative $n \to -\infty$ and the inner radius vanishes, then it has an *essential singularity* at $z_0$. According to *Picard's theorem*, essential singularities have the nasty property that $f[z]$ takes any, hence all, values in any arbitrarily small neighborhood infinitely often with possibly one exception. For example,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \tag{1.228}$$

has an essential singularity at the origin. The equation $w == e^{1/z}$ is satisfied by

$$w == e^{1/z} \Longrightarrow z = \frac{1}{\mathrm{Log}[w]} = (\mathrm{Log}\,|w| + i\,\mathrm{Arg}[w] + 2n\pi i)^{-1} \tag{1.229}$$

for any integer $n$. By choosing $n$ sufficiently large, one can make $|z|$ as small as desired. Thus, although $e^{1/z} \neq 0$, the one exception, all other values of $w$ are obtained infinitely often in a neighborhood of $z \to 0$, as expected from Picard's theorem.

Singularities in $f[z]$ at $z = \infty$ are classified according to the behavior of $f[\frac{1}{z}]$ at $z = 0$. Thus, $e^z$ has an essential singularity at $\infty$, while $z^{-n}$ is analytic at $\infty$ if $n$ is a positive integer.

### 1.13.4 Poles and Residues

Although the Laurent coefficients are defined in terms of an integral, it is usually easier to compute the coefficients using a derivative formula similar to that for the Taylor series. If $f[z]$ is analytic near $z_0$ except for an isolated $m$-pole at $z_0$, we define an auxiliary function

$$\phi[z] = (z - z_0)^m f[z] = \sum_{n=-m}^{\infty} a_n (z - z_0)^{m+n} \tag{1.230}$$

that is analytic within $|z - z_0| < R$ where $R$ is the radius of convergence for the Laurent series. The coefficients can then be obtained by differentiation, whereby

$$a_n = \frac{1}{(m+n)!}\left(\frac{d^{m+n}}{dz^{m+n}}\phi[z]\right)_{z=z_0} \tag{1.231}$$

This result can be written more succinctly as

$$a_n = \frac{\phi^{(m+n)}[z_0]}{(m+n)!} \tag{1.232}$$

where $\phi^{(k)}[z_0]$ denotes the $k^{\text{th}}$ derivative of $\phi[z]$ evaluated at $z_0$. This formula is similar to that for the Taylor series, except that $f$ is replaced by $\phi$ and the index is shifted, and reduces to the Taylor coefficients for an analytic function with $m = 0$. However, this method is not useful at an essential singularity where $m = \infty$.

Often we require only the residue of $f$ at $z_0$. For a simple pole we identify the residue as

$$m = 1 \Longrightarrow a_{-1} = \phi[z_0] = \lim_{z\to z_0}(z - z_0)f[z] \tag{1.233}$$

while for an $m$-pole one obtains

$$m > 1 \Longrightarrow a_{-1} = \frac{\phi^{(m-1)}[z_0]}{(m-1)!} \tag{1.234}$$

For example, consider

$$f[z] = \frac{z^n}{q[z]} = \frac{z^n}{az^2 + bz + c} \tag{1.235}$$

where we assume that $n \geq 0$, $a \neq 0$, and that $a, b, c$ are real. (Other cases can be treated separately.) The two poles at the roots of the denominator

$$z_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.236}$$

are distinct unless the discriminant $b^2 - 4ac$ happens to vanish. We can then write

$$f[z] = \frac{a^{-1}z^n}{(z - z_1)(z - z_2)} \Longrightarrow \rho_i = \frac{a^{-1}z_i^n}{z_i - z_j} \tag{1.237}$$

where $\rho_i$ is the residue at pole $z_i$ and $z_j$ is the other pole. If there happens to be a double pole, we find

$$z_1 = z_2 = z_0 \Longrightarrow a_{-1} = \left(\frac{d}{dz}(a^{-1}z^n)\right)_{z_0} = a^{-1}nz_0^{n-1} = a^{-1}n\left(\frac{-b}{2a}\right)^{n-1} \tag{1.238}$$

An important special case is provided by functions of the form

$$f[z] = \frac{p[z]}{q[z]} \quad \text{with} \quad q[z_0] = 0 \quad q'[z_0] \neq 0 \quad p[z_0] \neq 0 \tag{1.239}$$

where the simple pole at $z_0$ is a zero of $q[z]$. The Taylor expansion of $q[z]$ then takes the form

$$q[z] \approx q'[z_0](z - z_0) \tag{1.240}$$

such that the residue of $f[z]$ at $z_0$ becomes

$$a_{-1} = \frac{p[z_0]}{q'[z_0]} \tag{1.241}$$

For example, the function

$$f[z] = \frac{e^{az}}{e^z + 1} \Longrightarrow z_k = (2k+1)\pi i, \quad R_k = -e^{a z_k} \tag{1.242}$$

has poles at odd-integer multiples of $\pi i$ with residues easily determined using $q'[z_k] = -1$.

## 1.14 Meromorphic Functions

### 1.14.1 Pole Expansion

If a function $f[z]$ only has isolated singularities, it is described as *meromorphic*. For simplicity suppose that these singularities are simple poles at $z_n$ where the index lists the poles in order of increasing distance from the origin. The behavior near a simple pole can be represented by $z \approx z_n \Longrightarrow f[z] \approx \frac{b_n}{z - z_n}$. Thus, the function

$$g_n[z] = f[z] - \sum_{k=1}^{n} \frac{b_k}{z - z_k} \tag{1.243}$$

is analytic in a disk $|z| \le R_n$ where the radius $R_n$ encloses $n$ poles. According to the Cauchy integral formula, we may write

$$g_n[z] = \frac{1}{2\pi i} \oint_{C_n} \frac{g_n[s]}{s - z}\, ds = \frac{1}{2\pi i} \oint_{C_n} \frac{f[s]}{s - z}\, ds - \frac{1}{2\pi i} \sum_{k=1}^{n} b_k \oint_{C_n} \frac{ds}{(s - z)\left(s - z_k\right)} \tag{1.244}$$

where $C_n$ is a circle, $|z| = R_n$, that encloses $n$ poles without any poles being on the contour itself. For any $z \neq z_k$ we can use partial fractions to express the second contribution in a form

$$\oint_{C_n} \frac{ds}{(s - z)(s - z_k)} = \oint_{C_n} \frac{ds}{s - z_k} - \oint_{C_n} \frac{ds}{s - z} = 0 \tag{1.245}$$

where cancellation between equal residues is apparent. Furthermore, if $z = z_k$ we also find

$$\oint_{C_n} \frac{ds}{(s - z_k)^2} = 0 \tag{1.246}$$

and conclude that

$$g_n[z] = \frac{1}{2\pi i} \oint_{C_n} \frac{f[s]}{s - z}\, ds \tag{1.247}$$

for $z$ within $C_n$.

Next let

$$M_n = \max\left[\left|f\left[R_n e^{i\theta}\right]\right|\right] \tag{1.248}$$

represent the largest modulus found on the circle $C_n$, such that

$$\left|g_n[z]\right| \leq \frac{M_n R_n}{R_n - |z|} \tag{1.249}$$

bounds $g_n$. If $f$ is bounded such that $R_n \to \infty$ with finite $M_n$, we can construct a sequence of $g_n$ functions which are also bounded as $|z| \to \infty$. Thus, the function

$$g[z] = \lim_{n\to\infty} g_n[z] \tag{1.250}$$

is analytic and bounded in the entire complex plane. According to Liouville's theorem, such a function must be constant! Hence, we can write

$$f[z] = g_\infty + \sum_{k=1}^{\infty} \frac{b_k}{z - z_k} \tag{1.251}$$

and all that remains is to determine the value of the constant $g_\infty$. Using

$$f[0] = g_\infty - \sum_{k=1}^{\infty} \frac{b_k}{z_k} \Longrightarrow g_\infty = f[0] + \sum_{k=1}^{\infty} \frac{b_k}{z_k} \tag{1.252}$$

we finally obtain the *Mittag–Leffler theorem.*

**Theorem 17.** *Mittag–Leffler theorem: Suppose that the function $f[z]$ is analytic everywhere except for isolated simple poles, is analytic at the origin, and that there exists a sequence of circles $\{C_k : |z| = R_k, k = 1, n\}$ where each $C_k$ encloses k poles within radius $R_k$. Furthermore, assume that on these circles $|f|$ is bounded as $R_n \to \infty$. The function can then be expanded in the form*

$$f[z] = f[0] + \sum_{n=1}^{\infty} \left( \frac{b_n}{z - z_n} + \frac{b_n}{z_n} \right) \tag{1.253}$$

*where $b_n$ is the residue for pole $z_n$.*

Unlike Laurent expansions for which the choice of $z_0$ can be somewhat arbitrary, the *pole expansion* for meromorphic functions depends only upon intrinsic properties of the function itself. Although the present version places significant restrictions on the function, generalizations can often be made fairly easily. For example, if $f[z]$ has a pole at the origin, one can apply the theorem to the closely related function $g[z] = f[z + z_0]$ where $z_0$ is any convenient point where $f[z]$ is analytic. Similarly, if $M_n \propto R_n^{m+1}$ for large $R_n$, one can employ an expansion of the form

$$f[z] = \sum_{k=1}^{m} f^{(k)}[0] \frac{z^k}{k!} + \sum_{n=1}^{\infty} \frac{b_n}{z - z_n} \left( \frac{z}{z_n} \right)^{m+1} \tag{1.254}$$

Poles of higher order can be accommodated also, but we forego detailed analysis here.

Pole expansions appear in many branches of physics. If $f[z]$ represents the response of a dynamical system to some driving force, the poles generally represent resonances or normal modes of vibration while the residues represent the coupling of the driving force to those normal modes. Pole expansions can also be used to sum infinite series.

### 1.14.2 Example: Tan[$z$]

The function Tan[$z$] has simple poles at $z_n = (n + \frac{1}{2})\pi$ with residue $b_n = -1$ for integer $n$, both positive and negative. Thus, circles $C_n$ of radius $R_n = n\pi$ enclose $2n$ poles without singularities on the contours. One can show that $M_n \to 1$ as $n \to \infty$. Hence, Tan[$z$] fulfills all requirements for application of the Mittag–Leffler theorem. The pole expansion can now be expressed as

$$\mathrm{Tan}[z] = -\sum_{n=-\infty}^{\infty} \left( \frac{1}{z - \left(n + \frac{1}{2}\right)\pi} - \frac{1}{\left(n + \frac{1}{2}\right)\pi} \right) \tag{1.255}$$

$$= -\sum_{n=0}^{\infty} \left( \frac{1}{z - \left(n + \frac{1}{2}\right)\pi} - \frac{1}{\left(n + \frac{1}{2}\right)\pi} + \frac{1}{z + \left(n + \frac{1}{2}\right)\pi} + \frac{1}{\left(n + \frac{1}{2}\right)\pi} \right) \tag{1.256}$$

such that

$$\mathrm{Tan}[z] = \sum_{n=0}^{\infty} \frac{2z}{\left(\left(n + \frac{1}{2}\right)\pi\right)^2 - z^2} \tag{1.257}$$

With the substitution $z \to s\pi/2$, we obtain the *partial fraction representation*

$$\frac{\pi}{4s}\,\mathrm{Tan}\left[\frac{s\pi}{2}\right] = \frac{1}{1 - s^2} + \frac{1}{9 - s^2} + \frac{1}{25 - s^2} + \cdots \tag{1.258}$$

Expressions of this type can often be used to sum infinite series. For example, from

$$\lim_{s\to 0} \frac{\pi}{4s}\,\mathrm{Tan}\left[\frac{s\pi}{2}\right] = \frac{\pi^2}{8} \tag{1.259}$$

one immediately obtains

$$\sum_{k=0}^{\infty} \left(\frac{1}{2k+1}\right)^2 = \frac{\pi^2}{8} \tag{1.260}$$

Then using

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2k}\right)^2 + \sum_{k=1}^{\infty} \left(\frac{1}{2k+1}\right)^2 \Longrightarrow \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1}\right)^2 \tag{1.261}$$

we find

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1.262}$$

### 1.14.3 Product Expansion

If $f[z]$ is an entire function, its logarithmic derivative $\phi[z] = f'[z]/f[z]$ is a meromorphic function with poles at the roots of $f[z]$. If these roots are simple, the corresponding poles in $\phi$ will also be simple. Near a simple root we express $f[z]$ in the form

$$z \approx z_n \Longrightarrow f[z] \approx (z - z_n)p_n[z] \Longrightarrow \phi[z] \approx \frac{1}{z - z_n} \tag{1.263}$$

where $p_n[z]$ is smooth and nonvanishing near $z_n$. Hence, the poles of the logarithmic derivative all have residue $b_n = 1$. Provided that $\phi$ is suitably bounded at $\infty$, we can now use the pole expansion of $\phi$ to write

$$\frac{d\mathrm{Log}[f]}{dz} = \phi_0 + \sum_{n=1}^{\infty}\left(\frac{1}{z - z_n} + \frac{1}{z_n}\right) \tag{1.264}$$

$$\Longrightarrow \mathrm{Log}[f[z]] - \mathrm{Log}[f[0]] = z\phi_0 + \sum_{n=1}^{\infty}\left(\mathrm{Log}[z - z_n] - \mathrm{Log}[-z_n] + \frac{z}{z_n}\right) \tag{1.265}$$

where $\phi_0 = f'[0]/f[0]$. Exponentiating and simplifying this expression, we obtain the *product expansion*

$$\frac{f[z]}{f[0]} = \mathrm{Exp}\left[z\frac{f'[0]}{f[0]}\right]\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n} \tag{1.266}$$

where the $z_n$ are the roots of $f[z]$. Like the pole expansion of meromorphic functions, the product expansion of entire functions depends only upon intrinsic properties of the function. Expansions of this type are often useful in symbolic manipulations, but generally converge too slowly to be useful for numerical evaluations.

### 1.14.4 Example: Sin[$z$]

Although Sin[$z$] is an entire function with simple poles at $z_n = \pm n\pi$, we cannot employ the product expansion directly because $\phi_0$ is not finite. However, this difficulty is easily circumvented by considering instead the function

$$f[z] = \frac{\mathrm{Sin}[z]}{z} \Longrightarrow \phi[z] = \mathrm{Cot}[z] - \frac{1}{z}, \quad \phi[0] = 0 \tag{1.267}$$

The positive and negative roots can be accommodated by using two products

$$\frac{\mathrm{Sin}[z]}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{\frac{z}{n\pi}}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n\pi}\right)e^{-\frac{z}{n\pi}} \tag{1.268}$$

and combining factors pairwise to obtain

$$\frac{\mathrm{Sin}[z]}{z} = \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{n\pi}\right)^2\right) \Longrightarrow \mathrm{Sin}[z] = z\prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{n\pi}\right)^2\right) \tag{1.269}$$

This form displays all the roots of Sin[$z$] and is, in effect, a completely factored representation of its Taylor series.

## Problems for Chapter 1

**1. Complex number field**

In mathematics, a *field* $\mathbb{F}$ is defined as a set containing at least two elements on which two binary operations, denoted addition (+) and multiplication (×) satisfy the following conditions:

a) completeness and uniqueness of addition: $\forall a, b \in \mathbb{F}$, $c = a + b \in \mathbb{F}$ is unique

b) commutative law of addition: $a + b = b + a$

c) associative law of addition: $(a + b) + c = a + (b + c)$

d) $a + c = b + c \Longrightarrow a = b$

e) existence of identity element for addition: $\forall a, b \in \mathbb{F}, \exists x \ni a+x = b \Longrightarrow \exists 0 \ni a+0 = a$

f) completeness and uniqueness of multiplication: $\forall a, b \in \mathbb{F}$, $c = a \times b \in \mathbb{F}$ is unique

g) commutative law of multiplication: $a \times b = b \times a$

h) associative law of multiplication: $(a \times b) \times c = a \times (b \times c)$

i) $a \times c = b \times c \wedge c \neq 0 \Longrightarrow a = b$

j) existence of identity element for multiplication: $\forall a, b \in \mathbb{F}, \exists x \neq 0 \ni a \times x = b \Longrightarrow \exists 1 \ni a \times 1 = a$

k) distributive law: $a \times (b + c) = a \times b + a \times c$

The real numbers $\mathbb{R}$ obviously form a field with respect to ordinary addition and multiplication, but it is not immediately obvious that the complex numbers $\mathbb{C}$ form a field with respect to the extended definitions of addition and multiplication. To demonstrate that $\mathbb{C}$ is a field, you must identify the identity elements for addition and multiplication and must verify that each of the 11 conditions set forth above is satisfied.

**2. Triangle inequalities**

Prove the triangle inequalities: $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$.

**3. Applications of de Moivre's theorem**

Show that

$$\mathrm{Cos}[n\theta] = \mathrm{Cos}[\theta]^n - \binom{n}{2}\mathrm{Cos}[\theta]^{n-2}\,\mathrm{Sin}[\theta]^2 + \binom{n}{4}\mathrm{Cos}[\theta]^{n-4}\,\mathrm{Sin}[\theta]^4 + \cdots \tag{1.270}$$

$$\mathrm{Sin}[n\theta] = \binom{n}{1}\mathrm{Cos}[\theta]^{n-1}\,\mathrm{Sin}[\theta] - \binom{n}{3}\mathrm{Cos}[\theta]^{n-3}\,\mathrm{Sin}[\theta]^3 + \cdots \tag{1.271}$$

**4. Lagrange's trigonometric identity**

Prove:

$$\sum_{k=0}^{n}\mathrm{Cos}[k\theta] = \frac{1}{2} + \frac{\mathrm{Sin}\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\,\mathrm{Sin}\left[\frac{\theta}{2}\right]} \tag{1.272}$$

Hint: first prove

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \tag{1.273}$$

**5. Quadratic formula**

a) Prove that

$$az^2 + bz + c == 0 \Longrightarrow z = \frac{(b^2 - 4ac)^{1/2} - b}{2a} \tag{1.274}$$

applies even when $a, b, c$ are complex. Why did we not use a ± sign in front of the square root?

b) Use the quadratic formula to determine all roots of the equation $\mathrm{Sin}[z] == 2$. (Hint: $\mathrm{Sin}[z] = \frac{1}{2i}(w - \frac{1}{w})$ where $w = e^{iz}$.)

**6. Assorted trigonometric equations with complex solutions**

Find all solutions to the following equations assuming that $a, b$ are real numbers and that $|a| > 1$, $|b| > 1$. Express your results in the form $z = x + iy$ where $x, y$ are real-valued expressions that do not involve trigonometric functions and be sure to consider all cases.

a) $\mathrm{Cos}[z] == a$

b) $\mathrm{Cos}[z] == bi$

**7. Series RLC circuit**

A circuit contains resistance $R$, inductance $L$, and capacitance $C$ in series with a generator of electromotive force $\mathcal{E}[t] = \mathcal{E}_0 \mathrm{Cos}[\omega t]$. Let $I[t]$ represent the current flowing in the circuit and $Q[t]$ the charge stored in the capacitor. It is useful to express the physical quantities

$$\mathcal{E}[t] = \mathrm{Re}\left[\hat{\mathcal{E}} e^{i\omega t}\right], \quad I[t] = \mathrm{Re}\left[\hat{I} e^{i\omega t}\right], \quad Q[t] = \mathrm{Re}\left[\hat{Q} e^{i\omega t}\right] \tag{1.275}$$

in terms of complex phasors $\hat{\mathcal{E}}$, $\hat{I}$, and $\hat{Q}$ that represent both the magnitudes and relative phases for sinusoidal time dependencies.

a) Use Kirchhoff's laws to derive a phasor generalization of Ohm's law, $\hat{\mathcal{E}} = \hat{I}\hat{Z}$, where the impedance $\hat{Z} = Ze^{i\phi}$ is generally complex. Express the modulus, $Z$, and the phase, $\phi$, of the complex impedance in terms of the real parameters of the circuit.

b) Show that the power averaged over a cycle is given by $\bar{P} = \frac{1}{2}\mathrm{Re}[\hat{I}\hat{\mathcal{E}}^*]$ and evaluate this quantity in terms of real parameters. Show that $\bar{P}[\omega]$ exhibits a resonance and determine its position and full width at half maximum (FWHM). Sketch $\bar{P}[\omega]$ and $\phi[\omega]$ together.

**8. Smith chart**

The complex impedance $Z = R + iX$ for an AC circuit is decomposed into resistive and reactive components, $R$ and $X$, where $R > 0$ and $-\infty < X < \infty$. Smith proposed a representation

$$W = u + iv = \frac{Z-1}{Z+1} \tag{1.276}$$

that maps the right half-plane for $Z$ onto the unit disk for $W$. Determine the mappings for lines of constant $R$ and lines of constant $X$. Sketch illustrative samples of each.

**9. Bilinear mapping**

Study the bilinear mapping

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \tag{1.277}$$

by determining the images in the $w$-plane of representative lines and circles in the $z$-plane.

**10. Component functions**

Develop explicit expressions for the real and imaginary components, $u[x, y]$ and $v[x, y]$ for the following functions of $z = x + iy$.

a) $f[z] = (z^2 - 1)^{1/2}$

b) $g[z] = (z - 1)^{1/2}(z + 1)^{1/2}$

**11. Inverse trigonometric functions**

Prove:

$$\arcsin[z] = -i \log\left[iz + \left(1 - z^2\right)^{1/2}\right] \tag{1.278}$$

$$\arccos[z] = -i \log\left[z + \left(z^2 - 1\right)^{1/2}\right] \tag{1.279}$$

$$\arctan[z] = \frac{i}{2} \log\left[\frac{i+z}{i-z}\right] \tag{1.280}$$

This can be done by expressing equations of the form $z == \text{Sin}[f]$ in exponential form, substituting $w = e^{if}$, solving for $w$, and deducing $f[z]$. Determine the branch cuts needed to specify the principal branch of each function.

**12. An identity**

Prove: $\text{ArcTan}\left[\frac{2z}{z^2-1}\right] = 2\,\text{ArcCot}[z]$

**13. Principal value for an imaginary power**

Suppose that

$$\eta = \left(\frac{ia-1}{ia+1}\right)^{ib} \tag{1.281}$$

where $a, b$ are real.

a) Show that this quantity is real and find a simple expression for its principal value.

b) Determine the position and magnitude of any discontinuities.

**14. Derivative wrt** $z^*$

Show that a function $f[x, y]$ of two real variables can be expressed as a function $g[z, z^*]$ of the complex variable $z = x + iy$ and its complex conjugate $z^* = x - iy$. Then show that the requirement $\partial g/\partial z^* == 0$ is equivalent to the Cauchy–Riemann equations for the components of $f$ and argue that an analytic function is truly a function of a single complex variable, instead of two real variables.

**15. Analyticity of conjugate functions**

Suppose that $f[z]$ is analytic in some region.

a) Under what conditions is $g[z] = f[z^*]$ analytic in the same region?

b) Under what conditions is $h[z] = f[z]^*$ analytic?

c) Under what conditions is $w[z] = f[z^*]^*$ analytic?

**16. Completion of analytic functions**

Which of the following functions $u[x, y]$ are the real parts of an analytic function $f[z]$ with $z = x + iy$? If $u[x, y] = \text{Re}\, f[z]$, determine $f[z]$.

a) $u = x^3 - y^3$

b) $u = x^2 - y^2 + y$

**17. Analyticity for the sum, product, quotient, or composition of two functions**

Suppose that $f_1[z] = u_1[x, y] + i\, v_1[x, y]$ and $f_2[z] = u_2[x, y] + i v_2[x, y]$ are analytic functions of $z = x + iy$. Show that $f_1 + f_2$, $f_1 f_2$, $f_1/f_2$, and $f_1[f_2[z]]$ are analytic functions under appropriate conditions by demonstrating consistency with the Cauchy–Riemann equations. Be sure to specify the requisite conditions for each case.

**18. Equipotentials and streamlines for exponential function**

Sketch the equipotentials $u[x, y]$ and streamlines $v[x, y]$ for $w = e^z$ where $z = x + iy$ and $w = u + iv$.

**19. Equipotentials and streamlines for** Tanh

Evaluate and sketch the equipotentials and streamlines for the hyperbolic tangent.

**20. Cauchy–Riemann equations in polar form**

Suppose that $z = x + iy = re^{i\theta}$ is expressed in polar form and let $f[z] = Re^{i\Theta}$ where $R[r, \theta]$ and $\Theta[r, \theta]$ are real functions of $r$ and $\theta$. Derive Cauchy–Riemann equations relating $\frac{\partial R}{\partial r}$ to $\frac{\partial \Theta}{\partial \theta}$ and $\frac{\partial R}{\partial \theta}$ to $\frac{\partial \Theta}{\partial r}$ for differentiable functions. (Hint: consider infinitesimal displacements $dz_r$ and $dz_\theta$ in the $\hat{r}$ and $\hat{\theta}$ directions.)

**21. Circular average of analytic function**

Demonstrate that if $f[z]$ is analytic within the disk $|z - z_0| \leq R$ then the average value of $f[z_0 + re^{i\theta}]$ on any circle $|z - z_0| = r < R$ is equal to the value at its center, $f[z_0]$.

**22. Maximum modulus principle**

Prove that, if $f[z]$ is analytic and not constant within a region $R$, then $|f[z]|$ does not have a maximum within the interior of $R$. Hence, if $f$ is analytic and not constant within $R$, $|f|$ must reach its maximum value on the boundary of $R$.

**23. Extrema of harmonic functions**

Suppose that $u[x, y]$ is harmonic and not constant within region $R$. Prove that $u[x, y]$ has no extrema (neither maximum nor minimum) within $R$; hence, its extrema must be found on the boundary of $R$. (Hint: apply the maximum modulus principle to $e^{f[z]}$ where $f$ is analytic within $R$.)

**24. Absence of extrema in $|f|$ for analytic functions**

If $f[z]$ is analytic in domain $D$, demonstrate that $|f[z]|$ has no extrema in $D$. Hint: use the Taylor series representation to show that no neighborhood $|z - z_0| < r$ contains an extremum.

**25. An application of the Cauchy integral formula**

Suppose that $f[z]$ is analytic on and within the simple closed positive contour $C$. Evaluate the following integrals.

a) $\frac{1}{2\pi i} \oint_C \frac{t f[t]}{t^2 - z^2} \, dt$

b) $\frac{1}{2\pi i} \oint_C \frac{t^2 + z^2}{t^2 - z^2} f[t] \, dt$

**26. Derivatives of analytic functions**

Assume that $f[z]$ is analytic on and within a positive simple closed contour $C$ that encloses $z$. Use induction to prove

$$f^{(n)}[z] = \lim_{\Delta z \to 0} \frac{f^{(n-1)}[z + \Delta z] - f^{(n-1)}[z]}{\Delta z} = \frac{n!}{2\pi i} \oint_C dw \frac{f[w]}{(w - z)^{n+1}} \tag{1.282}$$

where $f^{(n)}$ is the $n^{\text{th}}$ derivative of $f$.

**27. Fundamental theorem of integral calculus**

Prove the fundamental theorem of integral calculus: If $f[z]$ is analytic in a simply connected domain $D$ that includes $z_0$ and $z$, then

$$F[z] = \int_{z_0}^{z} f[t] \, dt \tag{1.283}$$

is also analytic in $D$ and $f[z] = dF[z]/dz$.

**28. Poisson integral formula**

a) Suppose that $f[z]$ is analytic within the disk $|z| = r \leq a$. Prove

$$f[z] = \frac{1}{2\pi i} \oint_C \left( \frac{f[s]}{s - z} - \frac{f[s]}{s - \frac{a^2}{z^*}} \right) ds \tag{1.284}$$

where $C$ is a circle of radius $a$ centered on the origin. Then deduce the *Poisson integral formula*

$$f\left[re^{i\theta}\right] = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f[ae^{i\phi}]}{a^2 + r^2 - 2ar\,\mathrm{Cos}[\phi - \theta]}\, d\phi \tag{1.285}$$

b) Suppose that we know the electrostatic potential $\psi$ on the surface of a long cylinder as the real part of an analytic function (consider only two spatial dimensions). Obtain a general formula for the potential at any point within the cylinder. Compare the potential at the origin with the mean-value theorem. Note that, although a formal proof is not required, the Poisson integral formula can be applied for any function that is harmonic within $C$ except for a finite number of jump discontinuities upon $C$.

c) As a specific illustration, compute the interior potential given that $\psi[ae^{i\phi}]$ has the constant value $V$ for $\phi_1 \le \phi \le \phi_2$ and is zero on the rest of the cylinder. Display the angular dependence for a representative selection of $r$ values for some choice of $\phi_2 - \phi_1$.

**29. Uniform convergence of power series**

Suppose that the power series $f_n[z] = \sum_{k=0}^{n} a_k z^k$ converges absolutely such that $f[z] = \lim_{n\to\infty} f_n[z]$ for $|z| < R$. Show that $f_n[z]$ converges uniformly in any subdisk $|z| \le B < R$.

**30. Convergence of series representation for $e^z$**

Demonstrate explicitly that the series $\sum_{k=0}^{\infty} z^k/k!$ is absolutely convergent for all $z$ and that it is uniformly convergent in $|z| \le R$ for any finite $R$. Can one properly claim uniform convergence for all $z$?

**31. Sharpened ratio test**

a) The integral test can be used to established absolute convergence of the series representation of the Riemann zeta function

$$\zeta[z] = \sum_{n=1}^{\infty} n^{-z} \tag{1.286}$$

when $\mathrm{Re}[z] > 1$. Use the Weierstrass theorem to prove that $\zeta[z]$ is analytic for $\mathrm{Re}[z] > 1$. (This function has an important role in number theory and often appears in theoretical physics. It can be extended to most of the complex plane by analytic continuation, but that is beyond the scope of this problem.)

b) Use this result to obtain a sharpened form of the ratio test that states when the ratio of successive terms takes the form

$$\left|\frac{a_{n+1}}{a_n}\right| \simeq 1 - \frac{s}{n} \tag{1.287}$$

for large $n$, the series converges absolutely if $s > 1$.

c) Prove the existence of Euler's constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \mathrm{Log}[n]\right) \tag{1.288}$$

**32. Laurent series**

For each of the following functions, construct a complete set of Laurent series about the specified point and specify their convergence regions.

a) $f[z] = \frac{1}{(z-1)(z-2)}$ about $z = 0$

b) $f[z] = \frac{2z}{z^2-1}$ about $z = 2$

c) $f[z] = \frac{1}{(z^2-1)^{1/2}}$ about $z = 0$

d) $f[z] = \mathrm{Sin}\left[z + \frac{1}{z}\right]$ about $z = 0$

**33. Laurent expansion for $z^2\,\mathrm{Log}\left[\frac{z}{1-z}\right]$**

a) Define a single-valued branch for

$$f[z] = z^2\,\mathrm{Log}\left[\frac{z}{1-z}\right] \tag{1.289}$$

and specify the region where your definition is real.

b) What is the nature of the singularity at infinity?

c) Construct a Laurent expansion for $|z| > 1$.

**34. Some trigonometric series based upon a Laurent series**

Evaluate the Laurent series for $(z-a)^{-1}$ where $-1 < a < 1$ in the region $|z| > a$. Then use $z \to e^{i\theta}$ to compute $\sum_{m=1}^{\infty} a^m\,\mathrm{Cos}[m\theta]$ and $\sum_{m=1}^{\infty} a^m\,\mathrm{Sin}[m\theta]$.

**35. Grounded cylinder normal to uniform external field**

a) Suppose that an infinitely long conducting cylinder of radius $a$ is grounded. The cylinder is subjected to a uniform external electric field directed perpendicular to its symmetry axis. Use an analytic function to evaluate and sketch the equipotential surfaces and the net electric field. (Hint: expand $\Phi[z] = \phi[z] + i\psi[z]$ as a Laurent series around the origin and determine the coefficients using the appropriate boundary conditions.)

b) A two-dimensional incompressible fluid flows around an infinite cylinder whose axis is normal to the plane of motion. At large distances the velocity field is uniform. Evaluate and sketch the streamlines near the cylinder using an analytic function.

**36. Isolated singularities**

Classify the isolated singularities for each of the following functions. Be sure to consider the point at $\infty$, using $z \to 1/w$ with $w \to 0$.

a) $\dfrac{z^2}{1+z}$

b) $\dfrac{1-\mathrm{Cos}[z]}{z}$

c) $z e^{1/z}$

d) $\dfrac{e^{iz}}{z^2+\Lambda^2}$

**37. Singularity sequence**
Identify and classify the singularities of

$$f[z] = \frac{1}{\mathrm{Sin}[1/z]} \tag{1.290}$$

Is the singularity at the origin isolated? Is it a branch point?

**38. Residues**
Locate the poles for each of the following functions and evaluate their residues.

a) $\dfrac{z+1}{z^2(z+2i)}$

b) $\mathrm{Tanh}[z]$

c) $\dfrac{e^z}{z^2+\pi^2}$

d) $\dfrac{1}{z^n\,(e^z-1)}$ (integer $n$)

**39. Pole expansions**
Develop pole expansions for the following functions, being sure to verify that the necessary conditions are satisfied.

a) $\mathrm{Cot}[z]$

b) $\mathrm{Csc}[z]$

**40. Product expansions**
Develop product expansions for the following functions, being sure to verify that the necessary conditions are satisfied.

a) $\mathrm{Cos}[z]$

b) $\mathrm{Sinh}[z]$

**41. Product expansion for even functions**
Suppose that $f[z]$ is entire and is even, such that $f[-z] = f[z]$, and that its roots are all simple. Also assume that, except for simple poles, its logarithmic derivative is bounded at infinity such that the product expansion of $f[z]$ converges.

a) Show that the product expansion can be expressed in the form

$$f[z] = f[0]\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right) \tag{1.291}$$

where $f[\pm z_n] = 0$ and where the product includes only one member of each pair of roots.

b) Show that

$$\frac{f''[0]}{f[0]} = -2\sum_{n=1}^{\infty}\frac{1}{z_n^2} \tag{1.292}$$

$$\frac{f^{(4)}[0]}{f[0]} = 3\left(\frac{f''[0]}{f[0]}\right)^2 - 12\sum_{n=1}^{\infty}\frac{1}{z_n^4} \tag{1.293}$$

c) Apply these results to $f[z] = \mathrm{Sin}[z]/z$ and evaluate the following sums:

$$\sum_{n=1}^{\infty} n^{-2} \qquad \sum_{n=1}^{\infty} n^{-4} \tag{1.294}$$

**42. Contour integration of logarithmic derivative**

Suppose that $f[z]$ is analytic within a domain $D$ containing the positive simple closed contour $C$. The function $\phi[z] = f'[z]/f[z]$ is known as the logarithmic derivative of $f$. Let

$$I = \frac{1}{2\pi i}\oint_C \phi[z]\, dz \tag{1.295}$$

a) Suppose that $z_0$ is the only zero of $f$ within $D$ and is of order $m$. Show that $I = m$ if $C$ encloses $z_0$.

b) Evaluate $I$ assuming that $f'[z] \neq 0$ in $D$ and that $C$ encloses $N$ roots of $f$ but that $f[z] \neq 0$ on $C$.

**43. Argument principle**

a) Suppose that $f[z]$ is analytic and nonzero on the positive simple closed contour $C$ and that it is meromorphic in the domain $D$ contained within $C$. The function $\phi[z] = f'[z]/f[z]$ is known as the logarithmic derivative of $f$. Prove that

$$\frac{1}{2\pi i}\oint_C \phi[z]\, dz = N_0 - N_p \tag{1.296}$$

where $N_0$ is the number of zeros and $N_p$ is the number of poles in $D$ where each accounts for multiplicity (e.g., a double root or double pole counts twice).

b) Show that

$$\oint_C \phi[z]\, dz = i\Delta_C \arg[f] = 2\pi i(N_0 - N_p) \tag{1.297}$$

measures the change in the argument of $f[z]$ as $z$ moves around $C$. (Hint: consider the image $C \to \Gamma$ under the mapping $w = f[z]$.)

**44. Rouchés theorem**

Prove that if $f[z]$ and $g[z]$ are both analytic on and within the simple closed contour $C$ and $|g[z]| < |f[z]|$ on $C$, then $f[z]$ and $f[z] + g[z]$ have the same number of zeros within $C$.