1
Introduction to the Theory of Oscillations

1.1
General Features of the Theory of Oscillations

Oscillatory processes and systems are so widely distributed in nature, technology, and society that we frequently encounter them in our everyday life and can, apparently, formulate their basic properties without difficulty. Indeed, when we hear about fluctuations in temperature, exchange rates, voltage, a pendulum, the water level, and so on, we understand that it is in relation to processes in time or space, which have varying degrees of repetition and return to their original or similar states. Moreover, these base properties of the processes do not depend on the nature of systems and Can, therefore, be described and studied from just the point of view of a general interdisciplinary approach. This is exactly the approach that the theory of oscillations explores, the subject of which are the oscillatory phenomena and the processes in systems of different nature. The theory of oscillations gets its oscillatory properties from the analysis of the corresponding models. As a result of such an analysis, a connection between the parameters of the model and its oscillatory properties is established.

The theory of oscillations is both an applied and fundamental science. The applied character of the theory of oscillations is determined by its multiple applications in physics, mechanics, automated control, radio engineering and electronics, instrumentation, and so on. In these spheres of science, a large amount of research of different systems and phenomena was carried out, using the methods of the theory of oscillations. Furthermore, new technical directions have arisen on the basis of the theory of oscillations, namely, vibrational engineering and vibrational diagnostics, biomechanics, and so on. The fundamental characteristic of the theory of oscillations is based on the studied models themselves. They are the so-called dynamical systems, with the help of which one can describe any determinate evolution in time or in time and space. It is exactly the study of dynamical systems that allowed the theory of oscillations to introduce the concepts and conditions, develop the methods, and achieve the results that exert a large influence on other natural sciences. Here, we only mention the linearized stability theory, the concept of self-sustained oscillations and resonance, bifurcation theory, chaotic oscillations, and so on.

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1.2 Dynamical Systems

Consider the system, the state of which is determined by the vector $\mathbf{x}(t) \in \mathrm{R}^n$. Assume that the evolution of the system is determined by a single parameter family of operators $G^t, t \in \mathrm{R}$ (or $t \in \mathrm{R}_+$) or $t \in \mathrm{Z}$ (or $t \in \mathrm{Z}_+$), such that the state of the system at the instant t

$$\mathbf{x}(t, \mathbf{x}_0) = G^t \mathbf{x}_0 \tag{1.1}$$

where $\mathbf{x}_0$ is its initial state (initial point). We also assume that the evolutionary operators satisfy the following two properties, which reflect the deterministic character of the described processes.

The first property: G^0 is the identity operator, that is,

$$\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0, \tag{1.2}$$

for any $\mathbf{x}_0$. This property means that the state of the system cannot change spontaneously.

The second property of the evolutionary operators is

$$G^{t_1+t_2} = G^{t_1} \cdot G^{t_2} = G^{t_2} \cdot G^{t_1}, \tag{1.3}$$

that is,

$$\mathbf{x}(t_1 + t_2, \mathbf{x}_0) = \mathbf{x}(t_1, \mathbf{x}(t_2, \mathbf{x}_0)) = \mathbf{x}(t_2, \mathbf{x}(t_1, \mathbf{x}_0)) \tag{1.4}$$

According to (1.3), the system reaches the same final state, regardless of whether it does so within one time interval $t_1 + t_2$ or over several successive intervals t_1 and t_2, equal in sum to $t_1 + t_2$.

The combination of all initial points • or of all possible states of the system (in this case, $X = \mathrm{R}^n$) is called a *phase space*, and a pair $(X, \{G^t\})$, where a family of evolutionary operators satisfies the conditions (1.2) and (1.3), is a dynamical system.

Dynamical systems are divided into two important categories, one with continuous time if $t \in \mathrm{R}$ or R_+ and another with discrete time if $t \in \mathrm{Z}$ or Z_+.

The evolution of the system corresponds to the motion of the representation point in the phase space along the trajectory $\Gamma = \bigcup_t G^t \mathbf{x}_0$. The family $\Gamma^+ = \bigcup_{t \geq 0} G^t \mathbf{x}_0 \left(\Gamma^- = \bigcup_{t<0} G^t \mathbf{x}_0 \right)$ is called a positive semi-trajectory going through the initial point $\mathbf{x}_0$. If the family$\{G^t\}$ is continuous at t (for dynamical systems with continuous time), then the trajectories (semi-trajectory) represent continuous curves at X. For the dynamical systems with discrete time, the trajectories are discrete subsets in the phase space.

Let us introduce the idea of the invariance of a set, which will be necessary in what follows. The set $A \subset X$ is called positively (negatively) invariant if it consists of positive (negative) semi-trajectories, that is, A is positively (negatively) invariant if $G^t A \subset A, t > 0 (t < 0)$. The set A is called invariant if it is invariant both when positive and when negative.

1.2.1 Types of Trajectories

Let us define the main types of the dynamical system trajectories.

1) The point $\mathbf{x}_0$ is called a fixed point of a dynamical system if $G^t\mathbf{x}_0 = \mathbf{x}_0$ for all t (for systems with continuous time, such points are more often called *equilibrium points*).
2) The point $\mathbf{x}_0$ is called periodic if there exists $T > 0$, such that $G^T\mathbf{x}_0 = \mathbf{x}_0$ and $G^t\mathbf{x}_0 \neq \mathbf{x}_0$ for $0 < t < T$, and its corresponding trajectory $\bigcup\limits_{0\le t\le T} G^t\mathbf{x}_0$ of the dynamical system passing through this point is periodic. A periodic trajectory is a closed curve in the phase space of a dynamical system with continuous time or a set of T-periodic points for the dynamical systems with discrete time.
3) The point $\mathbf{x}_0$ is called nonwandering if for any open set $U \ni \mathbf{x}_0$ of this point and any $t_0 > 0$ there exists $t > t_0$, such that $G^t U \bigcap U \neq \emptyset$. The trajectory going through a nonwandering point is called a nonwandering trajectory.

There is a correspondence between the trajectories of dynamical systems and the motions of real systems. Stationary states of real systems correspond to fixed points of dynamical systems, periodic motions correspond to periodic trajectories, and the motions with some degree of repetition of their states in time correspond to nonwandering trajectories.

Note that the aforementioned trajectories can also exist in the dynamical systems whose phase space is not necessarily R^n. For example, the phase space of a dynamical system describing the oscillations of a mathematical pendulum is a cylinder, $X = \mathrm{S}^1 \times \mathrm{R}$, as the state of the pendulum at any moment of time is uniquely described by its phase $\varphi(t)$ determined with accuracy up to $2\pi(\varphi \in \mathrm{S}^1)$ and by the value of its velocity $\dot{\varphi} \in \mathrm{R}$.

1.2.2 Dynamical Systems with Continuous Time

For many dynamical systems with continuous time, the rule, which allows one to find the state at any point in time according to the initial state, is shown by the following system of ordinary differential equations:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_N), \quad i = 1, 2, \ldots, N$$

or, in vector form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \mathbf{x} \in \mathrm{R}^n, \mathbf{F} : \mathrm{R}^n \to \mathrm{R}^n, \tag{1.5}$$

for which the conditions of existence and uniqueness of the solutions are satisfied (hereafter we denote differentiation in time by an overdot). In this case, the family $G^t\mathbf{x}_0$ is simply given by the solution of system (1.5) with the initial condition $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. For example, for the linear system

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where A is an $n \times n$ matrix with constant elements, the solution has the form $\mathbf{x}(t, \mathbf{x}_0) = e^{At}\mathbf{x}_0$, where e^{At} is an $n \times n$ matrix. As the matrices e^{At_1} and e^{At_2} commute for any pair t_1, t_2, the property (1.3)

$$e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = e^{At_2} \cdot e^{At_1}$$

is fulfilled. Evidently, the property (1.2) is also fulfilled.

In another example, we consider the system given in polar coordinates

$$\dot{\rho} = \lambda\rho, \quad \dot{\varphi} = \omega,$$

where ρ and ω are the parameters. The solution of this system has the following form:

$$\rho = \rho_0 e^{\lambda t}, \quad \varphi = \omega t + \varphi_0$$

Hence, the evolution operators are specified as follows:

$$G^t : (\rho_0, \varphi_0) \to (\rho_0 e^{\lambda t}, \omega t + \varphi).$$

Evidently, the properties (1.2) and (1.3) are fulfilled.

Note that the right-hand side of system (1.5) does not depend on time explicitly. Such systems are called *autonomous*. There is also a large number of problems (e.g., systems subjected to an alternating external force), which are described by dynamical systems whose right-hand sides depend on time explicitly. They are called *nonautonomous*.

1.2.3 Dynamical Systems with Discrete Time

Dynamical systems with discrete time are usually defined as follows:

$$\mathbf{x}(n+1) = \mathbf{F}(\mathbf{x}(n)), \tag{1.6}$$

where $\mathbf{F} : \mathrm{R}^n \to \mathrm{R}^n$ is the map and $n \in \mathrm{Z}_+ = \{0, 1, 2, ...\}$ is the discrete time.

For such systems, a trajectory is a finite or countable set of points in R^n. Another equivalent notation is also used sometimes for a dynamical system with discrete time:

$$\overline{\mathbf{x}} = \mathbf{F}(\mathbf{x}),$$

where $\overline{\mathbf{x}}$ is the image of the point **å** under the action of the map $\mathbf{F}$. In this manual, we will use both forms of notation of maps.

Let us illustrate the concept of a dynamical system with discrete time by using the example of a one-dimensional map,

$$\overline{x} = 2x, \ \mathrm{mod}\ 1 \tag{1.7}$$

The phase space of this map is the interval $[0, 1]$. Let $x(0) = 1/5$. Directly from (1.7), we obtain

$$x(0) = \frac{1}{5} \to x(1) = \frac{2}{5} \to x(2) = \frac{4}{5} \to x(3) = \frac{3}{5} \to x(4) = \frac{1}{5}$$

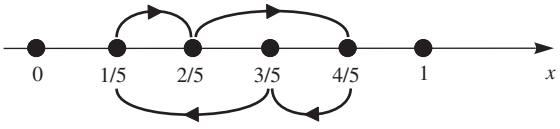

Figure 1.1 Semi-trajectory of system (1.7) with the initial condition $x(0) = 1/5$.

Hence, the considered semi-trajectory is periodic, with period 4 (see Fig. 1.1). At first sight, it seems that with such a simple rule of the map given by (1.7), the time evolution of the variable $x(n)$ under any initial conditions can be only simple and predictable. It turns out that it is not so. If the value of $x(0)$ is not known exactly, but within a certain precision ε, then the future behavior of $x(n)$ cannot be predicted. After a sufficiently large number of iterations, the interval $J_\varepsilon = (x(0) - \varepsilon, x(0) + \varepsilon)$ will cover the entire phase space, namely, the interval $[0, 1]$. In other words, there are trajectories going through the initial points in J_ε and reaching an arbitrary area of the phase space. The nonpredictability is caused here by the instability of the trajectories. This is the so-called deterministic chaos phenomenon, by which a priori nonpredictable motions appear in a deterministic system due to the instability of the trajectories.

1.2.4 Dissipative Dynamical Systems

Consider dynamical system (1.5) and introduce the concept of a dissipation ball. It is said that the smooth surface $S = \{\varphi(\mathbf{x}) = 0\}$ is transverse to the vector field $\mathbf{F}(\mathbf{x})$ if the scalar product

$$(\text{grad } \varphi(\mathbf{x}), \mathbf{F}(\mathbf{x})) \neq 0 \text{ for all } \mathbf{x} \in S,$$

where

$$\text{grad } \varphi = \left(\frac{\partial \varphi_1}{\partial x_1}, \frac{\partial \varphi_2}{\partial x_2}, \ldots, \frac{\partial \varphi_n}{\partial x_n} \right)$$

If S is a topological sphere, that is, the boundary of a topological ball D, then the ball D is called a dissipation ball provided that

$$(\text{grad } \varphi(\mathbf{x}), \mathbf{F}(\mathbf{x})) < 0 \text{ for all } \mathbf{x} \in S,$$

This means that the vector field $\mathbf{F}(\mathbf{x})$ on S is oriented into the interior of D (see Figure 1.2). Evidently, the trajectories within D will remain there for an indefinite time. Such dynamical systems are called dissipative. The main focus of this book will be on precisely such dynamical systems, which describe the processes in physical systems with various losses taken into account.

Definition 1.1. System (1.5) is called dissipative if there is a dissipation ball D, such that for any initial point $\mathbf{x}_0 \in \mathrm{R}^n$, $G^t\mathbf{x}_0 \in D$ for some $t > 0$.

Note that there are also other definitions of dissipative systems (e.g., sometimes it is required that div $\mathbf{F} < 0$, etc.), but we will be using Definition 1.1.

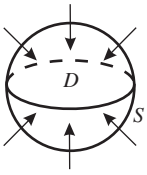

Figure 1.2 Qualitative representation of the dissipation ball D.

In the study of dissipative systems, the concept of the so-called absorbing region plays an important role.

Definition 1.2. A compact region D is said to be absorbing or trapping if

$$G^t D \subset \text{Int } D \text{ for } t > 0,$$

where Int D is the interior of D.

For example, for dynamical systems with discrete time of the form

$$\overline{x} = 3x(1 - x) = f(x)$$

the interval $[1/5, 4/5]$ is an absorbing region. Indeed, as

$$f\left(\frac{1}{5}\right) = f\left(\frac{4}{5}\right) = \frac{12}{25}, \quad f\left(\frac{1}{2}\right) = \frac{3}{4},$$

then

$$f\left(\left[\frac{1}{5}, \frac{4}{5}\right]\right) = \left[\frac{12}{25}, \frac{3}{4}\right] \subset \left(\frac{1}{5}, \frac{4}{5}\right).$$

1.3 Attractors

For systems with dissipation, it is very natural to differentiate between transition processes and established processes or regimes. The basic feature of an established process is that it "forgets" its initial state and is independent of it. This means that, after each finite time interval corresponding to the transition process, each positive semi-trajectory falls into a small vicinity of some invariant set – an attractor. There are several definitions of the attractor: a Milnor attractor, a statistical attractor, and so on. We take one of them, which, in our opinion, is most suitable for the aims of this book.

Definition 1.3. Let D be an absorbing region of the dynamical system (G^t, X). Then the set

$$A = \bigcap_{t \geq 0} G^t D$$

is called the *maximal attractor* in D.

Definition 1.4. The invariant set A is an attractor if there is an absorbing region D, for which A is the maximal attractor.

It is clear that the maximal attractor depends on the absorbing region and may contain other attractors. Examples of the simplest attractors are stable equilibria and fixed points.

1.4 Structural Stability of Dynamical Systems

It is evident that a dynamical system describing the behavior of any real system must depend on parameters. For example, we consider the system (1.5) which depends on some set of parameters

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu), \mu \in \mathrm{R}^{\mathrm{k}}, \tag{1.8}$$

where μ is the vector of the parameters. A question arises: "Is it possible to avoid using the methods of the theory of oscillations and to perform the required calculations of the dynamics of system (1.8) directly, for example, numerically, by using modern computers and numerical methods?" Suppose we can approximately construct a solution of the system with any initial conditions, assuming we have constructed some solution on a certain time interval. What can be said about the behavior of the entire system on the basis of the obtained information about of a single solution? Evidently nothing, as the initial conditions are almost always arbitrary in real systems. That is why the sampling of even a very large number of initial conditions does not solve the problem completely, because the behavior of the system under the remaining initial conditions remains unclear. Moreover, the problem is also complicated by the fact that the real systems depend on parameters. Therefore, by using numerical modeling, we can at best speak of the behavior of a real system only for some values of the parameters and some initial conditions.

Thus, for the construction of various devices and instruments to study the properties of real objects, it is necessary to examine not only one such particular solution of the system, but also *the entire class of models*. In order to solve this complex problem within the framework of the theory of oscillations, an approach is developed, which includes the following basic principles:

1) study not all trajectories of the system, but only the selected (in some sense, special) ones and look for parameters under which such trajectories exist;
2) study the behavior of the system trajectories for other values of the parameters, as a rule, only *qualitatively*.

It is evident that in the dynamical systems describing the motions of real systems, not any one of the factors considered can stay absolutely unchanged in time. Consequently, the dynamical systems, generally speaking, vary along with their parameters. However, if these variations are sufficiently small, then, as experience shows, the real system as if "ignores" these variations, that is, qualitative features of its behavior persist. That is why, if we wish to show this peculiarity for a dynamical system, we will need to assign to it the property of *roughness*. Namely, with small variations in the parameters, the qualitative structure of the partitioning of the phase space into trajectories should remain unchanged. By that we single out

the class of "rough" dynamical systems. The roughness of a dynamical system can be interpreted as the stability of the structure of partitioning its phase space into trajectories with respect to small variations of the dynamical system. That is why rough dynamical systems are often called structurally stable.

A.A. Anrdonov and L.S. Pontryagin (1937) introduced a rigorous mathematical definition of the roughness of dynamical systems with a two-dimensional phase space. Let us give it here for the system

$$\dot{x} = P(x, y), \dot{y} = Q(x, y) \tag{1.9}$$

where P and Q are smooth functions, and system (1.9) is dissipative with a dissipation ball D.

Definition 1.5. System (1.9) is called rough (structurally stable) if there is a small number $\delta > 0$ such that *all* dynamical systems of the type

$$\dot{x} = P(x, y) + p(x, y), \dot{y} = Q(x, y) + q(x, y), \tag{1.10}$$

where the analytical functions $p(x, y)$ and $q(x, y)$ satisfy the inequality

$$|p(x, y)| + |q(x, y)| + \left|\frac{\partial p}{\partial x}\right| + \left|\frac{\partial q}{\partial x}\right| + \left|\frac{\partial p}{\partial y}\right| + \left|\frac{\partial q}{\partial y}\right| < \delta,$$

have the same structure of partition of D into positive semi-trajectories as system (1.9).

It is absolutely clear that the roughness of a dynamical system does not persist with any variation of a parameter. A parameter can be varied in such a way that a fundamental change occurs in the phase portrait. One rough dynamical system converts into another via a nonrough dynamical system. The value of the parameter at which the dynamical system is nonrough is called a *bifurcation value*. The requirement of roughness for autonomous systems of the second order, while being natural from the point of view of applications, considerably simplifies the possible structures of partition of the phase plane into trajectories. Each of these structures is determined by a finite number of special phase trajectories. The specific nature of these trajectories will be discussed later in this book.

Note that the direct transfer of the aforementioned definition of roughness to the case of multidimensional (with the phase space of three or more dimensions) dynamical systems proved to be impossible. It was found that there are multidimensional systems containing only unstable trajectories, and in the space of dynamical systems, there are entire regions of nonrough systems. Therefore, the theory of rough multidimensional dynamical systems is constructed differently than in the two-dimensional case.

1.5 Control Questions and Exercises

1.1 Find 2- and 3-periodic trajectories of dynamical system (1.7).
1.2 Show that system $\dot{x} = x - x^3$ is dissipative.

1.3 Find the absorbing region for the map $\overline{x} = 3.1x(1 - x)$.

Show that each of the following systems is a dynamical system with dissipation.

1.4

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -y - x^3. \end{cases}$$

1.5

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases}$$

where $\sigma = 10, b = 8/3, r > 0$.

1.6

$$\begin{cases} \dot{\varphi} = y, \\ \dot{y} = -y - a\cos(\varphi), \end{cases}$$

where $a \neq 0$. Find the absorbing region.