1
Electromagnetic Radiation

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In the beginning ... darkness was upon the face of the deep.
And God said, Let there be light: and there was light.

Genesis 1:1-3

1.1 Introduction

Light has been a trusted probe of a variety of aspects of the universe since the beginning of scientific inquiry. Today, regardless of whether searching for gravitational waves, exploring the fundamental properties of quantum mechanics, or designing metamaterial, light continues to play a critical role in revealing nature and engineering tools to enhance life. In this chapter, we review a few key elements of classical and quantum light. A comprehensive review of light is well beyond the scope of this chapter. Thus, we have chosen to focus on the properties that are most often encountered in the laboratory while providing some context and history.

Light, an electromagnetic (EM) field, is an intimate coupling between time-dependent electric and magnetic fields. Classically, the EM field is described quantitatively through Maxwell's equations (Section 1.3.1), where it can be viewed as a wave[1)] – a disturbance – satisfying the wave equation,

$$\nabla^2 \Psi = \varepsilon\mu \frac{\partial^2 \Psi}{\partial t^2} \tag{1.1}$$

In Eq. (1.1), ε and μ are the permittivity and permeability of the medium through which the light is traversing ($\sqrt{1/\varepsilon\mu} = v$ is the speed of light or phase velocity in the medium), and ∇^2 is the Laplacian.[2)] The form of the solution depends on the coordinate system – rectangular, spherical, and so on – in which ∇^2 is expressed, as we discuss in Section 1.3.2. However, in general, the solution for the so-called running wave in one dimension is

$$\Psi(\mathbf{x}, t) = f(\mathbf{x} \pm vt) \tag{1.2}$$

1) Christiaan Huygens, a contemporary of Isaac Newton, viewed light as a wave prior to the mathematical formulation as we now know it. Newton, on the other hand, was convinced that light was a stream of corpuscles.

2) The Laplacian is shorthand for

$$\nabla \cdot \nabla$$

which can be written as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in rectangular coordinates.

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where **x** describes the distance that the disturbance moves as t increases. The "−"("+") sign indicates motion in the positive (negative) **x** direction.

In vacuum, $\varepsilon \to \varepsilon_0$, $\mu \to \mu_0$, and $v \to c = \sqrt{1/\varepsilon_0\mu_0}$, the vacuum light speed. Special relativity tells us that c sets a "speed boundary" across which information cannot flow. Specifically, those of us living in a "sub-c" universe are prohibited from achieving speeds equal to or larger than c as well as those living in a "super-c" universe from speeds lower than c. Light waves in vacuum are very special and differ from other waves that we encounter in everyday life. There is no rest frame for light and light travels at the same speed in all frames.

As is true of all waves, light is characterized by a wavelength (λ) and a frequency (ν). In vacuum, these quantities are linked by c,

$$\nu \equiv \frac{c}{\lambda} \tag{1.3}$$

In media different from vacuum $v = c/n$ where,

$$n = \sqrt{\left(\frac{\varepsilon}{\varepsilon_0}\right)\left(\frac{\mu}{\mu_0}\right)} \tag{1.4}$$

is the index of refraction and the quantities in parentheses are the electric and magnetic dielectric constants respectively; μ differs slightly for μ_0 for most cases of interest. The more general relationship between λ and ν is

$$\nu = \frac{v}{\lambda} = \frac{c}{\lambda_0} \tag{1.5}$$

where λ_0 is defined as the vacuum wavelength. By definition, ν is medium independent and maintains its vacuum value so

$$\lambda = \frac{\lambda_0}{n} \tag{1.6}$$

We note that it is possible for the speed of massive particles to exceed the speed of light in media. Shock waves that result are similar to a sonic boom for sound waves. In the case of light, it is called *Cherenkov radiation*.[3] The emitted light is confined to a cone, the half angle of which is defined by

$$\alpha = \cos^{-1}\left(\frac{c}{nv_p}\right) \tag{1.7}$$

where v_p is the particle speed.

The wavelength spectrum of light is vast, ranging from radio waves to γ-rays, with characteristic wavelengths as large as astrophysical objects to as small as nuclei, respectively (see Section 1.2). A narrow light source, such as a line-narrowed continuous wave (CW) laser, is often said to emit a "single frequency;" such light is termed *monochromatic light*. Monochromanicity, however, is a relative statement. Monochromatic as compared to what? All known sources of light emit within some bandwidth – a spread in wavelength ($\Delta\lambda$) or frequency ($\Delta\nu$) – be it as broad as the solar spectrum or as narrow as the resonance line of an atom.[4] While λ and ν are inversely proportional, it is helpful to

3) The 1958 Nobel Prize in Physics went to Pavel A. Cherenkov for his discovery in 1934 that bears his name, which he shared with Ilya M. Frank and Igor Y. Tamm for their explanation (Cherenkov, Frank and Tamm, 1958).

4) Microwave sources and state-of-the-art ultra-stable lasers can have widths of a fraction of a Hertz. Even still, $\Delta\nu \neq 0$!

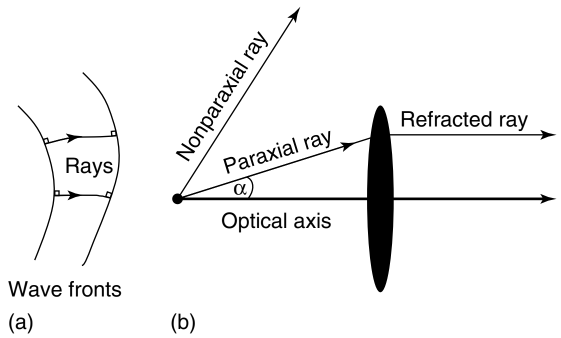

Fig. 1.1 Definition of rays and wave fronts (a) and paraxial rays (b) where $\alpha \ll 1$ rad.

recognize that

$$\left|\frac{\Delta\lambda}{\lambda}\right| = \left|\frac{\Delta\nu}{\nu}\right| \tag{1.8}$$

Classically, the treatment of light falls into two categories: geometrical and physical optics. In geometrical optics, the wave properties (e.g., diffraction) are ignored. Conceptually, we let $\lambda \to 0$ and instead discuss rays. While we do not review the usage of rays in this chapter, we do point out that ray tracing is employed extensively for designing optical systems. Rays are related to waves in that they are perpendicular lines joining the wave fronts (see Figure 1.1). The wave fronts turn out to be the surfaces of constant phase and so rays point in the direction of energy flow. Thus, rays from a point source are radial lines perpendicular to spherical surfaces. Typically, when dealing with rays, we focus on a subset of all the rays called *paraxial rays*. These rays are nearly parallel or form a small angle about a preferred direction. In the example shown in Figure 1.1, the ray traversing the center of the lens is the preferred direction and is called the *optical axis*. Paraxial rays deviate from the optical axis by such a small amount that $\sin\alpha \simeq \tan\alpha \simeq \alpha$. Rays are useful for describing refraction, the bending or redirection of light at the interface between two media with different indices of refraction, and reflection. Refraction, responsible for focusing of light by lenses and the angular spread of the $\Delta\lambda$ components after passing through prisms, is a result of momentum conservation and is succinctly stated through Fermat's principle: *light traverses a path from A to B that is an extremum of the optical path length (OPL).*[5] That is,

$$\delta(OPL) \equiv \delta\left(\int_A^B n(s)ds\right) = 0 \tag{1.9}$$

where s is the geometric path. Fermat's Principle leads to two important properties of light. First, the law of reflection,

$$\theta_i = \theta_r \tag{1.10}$$

5) The principle is often stated as the shortest path, that is, $\delta(OPL)$ would be a minimum. However, the calculus of variation only uses the fact that OPL is stationary; the second derivative is not considered. Thus, while usually the case, the path taken is not necessarily the minimum optical path.

where θ_i (θ_r) is the incident (reflected) angle. The second is Snell's Law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1.11}$$

relating the incident (θ_1) and refracted (θ_2) angles for a ray refracted (i.e., bent) as it passes through an interface between two media with different indices of refraction.

It is interesting to note that Fermat's principle (ca 1657) is closely related to Maupertu's principle in mechanics (ca 1744) for self-contained systems obeying conservation laws,

$$\delta\left(\int_A^B p ds\right) = 0 \tag{1.12}$$

where p is the momentum. Equation (1.12) is the principle of least action[6] when formulated more generally as

$$\delta\left(\int_A^B \mathcal{L}\, dt\right) = 0 \tag{1.13}$$

with $\mathcal{L}$ being the Lagrangian.[7] These equations show that particles and rays of light assume rectilinear motions in free space or when there are no forces (fields[8]) and the index of refraction is constant. In general, the trajectories of particles and light are stationary. The index of refraction plays the role of a field causing rays to deviate from linearity when not constant just like forces (potentials) cause particle trajectories to bend.

The geometrical approximation is good when the variation of the physical features of the media are large in comparison to λ. When they become comparable to λ, the wave properties of the light must be considered. The realm of physical optics allows descriptions of elements such as apertures and grating. It further provides a framework to discuss fundamental concepts such as diffraction (the angular spread of a beam of light and the bending of light around obstacles Section 1.7), interference (the superposition of two or more waves, leading to constructive and destructive sums depending on the relative phase of the waves Section 1.8), and coherence (issues associated with how stable the phase is in time and across wave fronts).

The smallest unit of light is called the *photon*, light quanta after the German *Lichtquanten* meaning portions of light.[9] While centuries before the age of quantum physics, Isaac Newton championed the idea of light as a stream of corpuscles, photons are quantum entities whose behavior under certain conditions are well known. However, the answer to the question "*What is a photon precisely*" continues to be illusive. Over the years, the *definitions* tend to fall into one of three distinct categories:

- a fundamental particle;
- an elementary excitation of the EM field; or
- something registered by a photodetector.

6) Like with Fermat's principle, least action is a bit of a misnomer; stationary action would be more appropriate as again only the first derivative is considered.

7) We point out that Hamilton, Lagrange, Euler and others played a role in the development of the principle as well.

8) Even in vacuum, the trajectory of light is deflected by a gravitational field. See, for example, Refs. Misner, Thorne and Wheeler (1973) and Hartle (2003) for a discussion of light in a gravitational field.

9) Gilbert N. Lewis is given credit for coining this name (Lewis, 1926).

In this chapter, we do not argue for or against one view over another.

Massless photons, like massive particles, carry both energy, $h\nu$, and momentum, h/λ, where h is Planck's constant. However, the photon wavefunction must be constructed with care. There have been suggestions that the photon can be understood as simply a classical field plus vacuum fluctuations[10] – a semiclassical approach if you will. There are cases, however, where such an approach gives the wrong answer (as determined by experimental observation). Thus, we have two regimes: classical light and quantum light. By definition, quantum light is any behavior of light that cannot be explained by classical fields, that is, solutions to the wave equation. An example would be squeezed light (Henry and Glotzer, 1988).

The photon is considered to be a fundamental particle. It has an intrinsic spin, which is an integer of unit magnitude. Thus, it obeys Bose statistics, but it has only two states of helicity (aligned or antialigned with its direction of propagation) because being massless, it has no vacuum rest frame. In addition to spin, light has other nonclassical features, typically revealing themselves through intensity noise, correlations, and counting statistics. Finally, both classical fields as well as photons can carry orbital angular momentum and support vortices and solitons (Desyatnikov, Kivshar and Torner, 2005; Kivsha and Agrawol, 2003; Pismen, 1999).

We conclude this introduction by pointing out that if we substitute h/λ for p into Eq. (1.12), we get a different formulation of Fermat's principle,

$$\delta\left(h\int_A^B \frac{ds}{\lambda}\right) = 0 \qquad (1.14)$$

The close analogy between Eqs (1.12) and (1.14) suggests an intimate connection between matter and light, from which one can postulate a wave equation for matter similar to that for light. As we discuss in Section 1.9, if we identify the wavelength of the particle as h/p (the de Broglie wavelength) and the index of refraction with $(U-V)/U$, where U and V are the total and potential energies, respectively, the time-independent Schrödiger equation emerges in a form that is not very different from the wave equation for light. Thus, photons, like matter, exhibit both wave and particle behavior.

With this overview as a backdrop, the remainder of this chapter is devoted to the details of selected characteristics of light. We start with the description of EM spectrum in Section 1.2 followed by a review of the wave equation and its solutions in Section 1.3. In Section 1.4, we consider radiometric issues and address the vector nature of light in Sections 1.5 and 1.6. We cover diffraction and interference in Sections 1.7 and 1.8 and conclude the chapter by further discussing the photon matter analogy in Section 1.9.

1.2 The Spectrum of Light

The EM spectrum is traditionally divided into the seven regions shown in Figure 1.2. It should be understood that the boundaries between these regions as well as those between subregions are

10) Vacuum fluctuations refer to the photons that are created spontaneously from the vacuum.

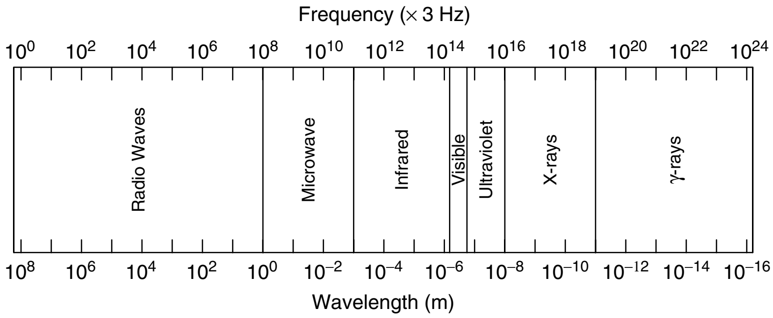

Fig. 1.2 The electromagnetic radiation spectrum.

not hard and fast, nor are the number of subregions unique. The most familiar region of the spectrum, the visible region, consists of wavelengths that range from about 0.40 μm at the blue end to 0.78 μm at the red end. Table 1.1 shows the corresponding colors for the wavelengths between. The visible subregions are a good example of the nonuniqueness of subbands of regions; for example, some references include cyan between green and blue while others insert indigo between blue and violet. Breaking the spectrum into six rather than seven or eight bands is of little consequence typically, because most objects emit a range of colors (i.e., $\Delta\lambda$ is relatively broad) or multiple colors (e.g., $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ again spanning a large $\Delta\lambda$), making the identification of a pure color a rare event. Of course, when $\Delta\lambda$ is small as it often is for some lasers, our eyes in fact do perceive a pure color. For example, consider the red Helium–Neon laser at 632.8 nm or the green doubled Nd : YAG laser at 532 nm.

Subbands also exist for the other regions of the EM spectrum. Tables 1.2–1.4 give some of the more familiar subbands for the other regions. More about the spectrum of light can be found in Ref. (HyperPhysics, 2006).

1.3 Basics of Electromagnetic Waves

As mentioned in the introduction, physical optics is concerned with the

Tab. 1.1 The approximate wavelength, frequency, and energy ranges for six primary visible color bands. Energies increase from left to right.

Color	**Wavelengths (nm)**	**Frequencies ($\times 10^{14}$ Hz)**	**Energies (eV)**
Red	780–625	3.8–4.8	1.6–2.0
Orange	625–590	4.8–5.1	2.0–2.1
Yellow	590–565	5.1–5.3	2.1–2.2
Green	565–500	5.3–6.0	2.2–2.4
Blue	500–435	6.0–6.9	2.4–2.8
Violet	435–380	6.9–7.9	2.8–3.6

Tab. 1.2 The approximate wavelengths, frequencies, and energies of key radio bands.

Designation	Wavelengths (m)	Frequencies (MHz)	Energy (meV)
AM radio	560–190	0.540–1.600	$(2.33–6.62) \times 10^{-3}$
TV	5.5–0.33	54–890	0.223–3.68
FM radio	3.40–2.78	88.1–108.1	0.364–0.447
Cell phone	0.43, 0.35, 0.18, 0.16, and 0.14	700, 850, 1700 1900, and 2100	2.89, 3.52, 7.03 7.85, and 8.68
Satellite radio	0.129–0.128	2320–2345	9.59 and 9.70
WiFi[a]	0.124–0.121	2412–2480	9.97–10.1
Radar	300–0.001	1–300 000	0.00414–1241

[a]The IEEE 802.11b/g/n standards communicate at 2.4 GHz. The IEEE 802.11a standard communicates at 5 GHz, but is essentially obselete.

Tab. 1.3 The approximate wavelength, frequency, and energy ranges of the three primary infrared (IR) bands.

Color	Wavelengths (μm)	Frequencies ($\times 10^{14}$ Hz)	Energies (eV)
Far IR	1000–10	3×10^{-3}–0.3	1.24×10^{-3}–0.124
IR-C	10–3.0	0.3–1.0	0.124–0.414
IR-B	3.0–1.4	1.0–2.14	0.414–0.886
IR-A	1.4–0.7	2.14–4.28	0.886–1.77

wave properties of light, which we examine in this section. We first review how Maxwell's equations lead to the wave equation for the components of the E and B fields. We examine EM wave solutions and their properties, primarily focusing on laser light in homogeneous media in the absence of charge and current sources.

1.3.1 Maxwell's Equations

As we alluded to in the introduction, a wave theory of light predates Maxwell's treatment. In the late 1800s, it was believed that the *aether*, an omnipresent elastic medium, supported light propagation much like the atmosphere or material supports sound waves. Although the theory based on the *aether* had consistency issues and required ad hoc assumptions to sustain it, its belief was so pervasive that even the null result of the experiments by Michelson and Morley could not easily dethrone it.[11)] It was not until Maxwell showed that his

11) The Michelson-Morley experiment was designed to measure a shift in interference fringes (see Section 1.8.1 for a discussion of interference) caused by a change in the speed of light moving with and perpendicular to the *aether*. The experiment is discussed in Refs. Michelson (1881) and Michelson and Morley (1887a,b).

Tab. 1.4 The approximate wavelength, frequency, and energy ranges of the three primary ultraviolet (UV) bands.

Designation	Wavelengths (nm)	Frequencies ($\times 10^{14}$ Hz)	Energies (eV)
UVA	400–320	7.45–9.37	3.10–3.88
UVB	320–280	9.37–10.7	3.88–4.43
UVC	280–200	10.7–15.0	4.43–6.2
Vacuum UV	200–50	15.0–60.0	6.2–24.8
Extreme UV	80–2.5	$37.5–1.2 \times 10^3$	15.5–500
Soft X-ray	4.5–0.15	$(0.7–2) \times 10^4$	$275–8.3 \times 10^3$

mathematical formulation of Gauss's, Ampere's and Faraday's laws (Maxwell's equations) lead naturally to an EM wave equation that could account for observed phenomena without resorting to arbitrary assumptions that the *aether* idea was abandoned.

Maxwell's equations in a medium with sources (represented by ρ, the charge density) and currents (represented by $\mathbf{j}$, the current density) can be written as

$$\nabla \cdot \mathbf{D} = \rho \tag{1.15}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{1.16}$$

$$\nabla \times \mathbf{E} = \mathbf{j} - \frac{\partial \mathbf{B}}{\partial t} \tag{1.17}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{1.18}$$

where

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{1.19}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{1.20}$$

These equations are simplified in the absence of sources, $\rho \to j \to 0$, and are the only cases we consider in this chapter.

1.3.2
Wave Equation

It is very straightforward to show that $\mathbf{E}$ and $\mathbf{H}$ of the EM field satisfy Maxwell's equations and the wave equation simultaneously in a homogeneous medium where $\nabla\varepsilon = \nabla\mu = 0$, by taking the curl of Eqs (1.17) and (1.18). Using the vector identity, $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2\mathbf{V}$, and the fact that $\nabla \cdot \nabla \times \mathbf{V} = 0$ leads to

$$\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.21}$$

$$\nabla^2 \mathbf{H} = \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} \tag{1.22}$$

which, by inspection, are the same as Eq. (1.1) because $\varepsilon\mu = n^2\varepsilon_0\mu_0 = 1/v^2$. Of course, the general solution is of the form of Eq. (1.2), however, the specifics depend on the geometry and constraints of the problem.

1.3.2.1 Plane Waves

The simplest solution for EM waves is that of a plane wave, where each component of $\mathbf{E}$ and $\mathbf{H}$ is a function of $\xi = \hat{u} \cdot \mathbf{r} - vt$. Recall that $\hat{u} \cdot \mathbf{r} =$ constant defines a plane with $\hat{u}$ being a dimensionless unit vector perpendicular to that plane. It is straightforward then to show that

$$\frac{\partial \mathbf{E}}{\partial t} = -v\frac{\partial \mathbf{E}}{\partial \xi} \tag{1.23}$$

$$\nabla \times \mathbf{E} = \hat{u} \times \frac{\partial \mathbf{E}}{\partial \xi} \tag{1.24}$$

and similarly for **H**. Because **E** and **H** must satisfy Eqs (1.15)–(1.18) (again assuming a homogeneous medium), we can further write

$$\hat{u} \times \frac{\partial \mathbf{E}}{\partial \xi} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\partial \mathbf{H}}{\partial \xi} \tag{1.25}$$

$$\hat{u} \times \frac{\partial \mathbf{H}}{\partial \xi} = -\sqrt{\frac{\varepsilon}{\mu}}\frac{\partial \mathbf{E}}{\partial \xi} \tag{1.26}$$

Integrating Eqs (1.25) and (1.26) and setting the constant to zero (no contribution from the background) leads to

$$\mathbf{E} = -\sqrt{\frac{\mu}{\varepsilon}} \quad \hat{u} \times \mathbf{H} \tag{1.27}$$

$$\mathbf{H} = \sqrt{\frac{\varepsilon}{\mu}} \quad \hat{u} \times \mathbf{E} \tag{1.28}$$

This implies that **E**, **H**, and $\hat{u}$ form a right-handed orthogonal triad and that light is a transverse field, that is, **E** and **H** oscillate in a plane normal to the propagation direction, in homogeneous media (including vacuum) without sources.

1.3.2.2 Scalar Harmonic Waves

The most common building block for the EM wave is a wave that is harmonic in both time and space. These exhibit sinusoidal variation. Typically, a scalar wave can be expressed as either a real quantity[12)]

$$\Psi(\mathbf{r}, t) = A(\mathbf{r}, t)\cos(\mathbf{k}\cdot\mathbf{r} \pm \omega t + \varphi) \tag{1.29}$$

or a complex quantity,

$$\Psi(\mathbf{r}, t) = A(\mathbf{r}, t)e^{i(\mathbf{k}\cdot\mathbf{r} \pm \omega t + \varphi)} \tag{1.30}$$

where A is an amplitude that is a slowly varying function of position and time (compared with the rapid variation of the sinusoidal arguments), **k** is the wavevector ($|\mathbf{k}| = 2\pi/\lambda$) and ω ($= 2\pi\nu$) is the angular frequency. Now it should be clear that $\hat{u}$. The harmonic time dependence of the wave allows the wave equation to be written as

$$\nabla^2\psi + k^2\psi = 0 \tag{1.31}$$

Because **E** and **B** are vectors, the EM wave is actually a vector wave. Generally, each component of the field satisfies the wave equation (Eq. 1.31) and has solutions like those in Eq. (1.29) and (1.30).

The argument of the harmonic wave consists of two phase terms, $\overline{\xi}_{\pm} \equiv \mathbf{k}\cdot\mathbf{r} \pm \omega t$, the dimensionless version of ξ, and φ. A constant $\overline{\xi}_{-}$ ($\overline{\xi}_{+}$) defines a profile or phase of the wave that moves toward more positive (negative) r as time evolves at a speed

$$v = \frac{\omega}{|\mathbf{k}|} \tag{1.32}$$

known as *the phase velocity*. The second phase term is often referred to as the *relative phase of the wave*. It can be a fixed constant or time dependent. When it is a constant or has a well defined time dependence, it gives rise

12) Note, we have multiplied ξ by $|\mathbf{k}|$ ($|\mathbf{k}|v = \omega$ because $|\mathbf{k}| = n\omega/c$) to make the argument dimensionless.

to coherence. When it varies randomly with time, the light is said to be incoherent. Furthermore, when $\varphi(t) = \text{const}\cdot t^n$, the frequency changes with time. A linear dependence simply shifts the frequency while higher powers chirp the frequency – as the wave passes the frequency either increases or decreases, depending on the sign of the constant. More complicated functions are possible as well.

1.3.2.3 Waves with Curved Phase Fronts

Although plane waves are highly convenient to use, they are appropriate only when dealing with light that is effectively far from its source.[13] Many situations do not fall into this category. It is beyond the scope of this chapter to discuss non-planar waves extensively, but we will give two examples. For a more extensive discussion, the reader is directed to the text by Cowan, 1968. First, when the fronts are not planes, the solutions in Eq. (1.30) must be modified to correspond to the Laplacian being expressed in a different coordinate system. For example, a spherical wave takes the form

$$\Psi(r,t) = \frac{A}{r} e^{i(kr \pm \omega t + \varphi)} \tag{1.33}$$

where kr = constant. The phase fronts are clearly spheres. A bit more complicated example would be a cylindrical wave, which follows

$$\Psi(\rho, z, \theta, t) = AJ_m(k\rho) e^{\pm i k_z z} e^{\pm i m\theta} e^{-i(\omega t + \varphi)} \tag{1.34}$$

where $J_m(k\rho)$ is the m^{th} order Bessel function of the first kind (which are regular at the origin)[14] with m being a positive integer, and

$$k^2 = \left(\frac{\omega}{c}\right)^2 + k_z^2 \tag{1.35}$$

The surfaces of constant phase are just cylinders in this case.

1.4 Energy, Intensity, Power, and Brightness

Because the EM field is composed of E and B fields, its *Energy Density* is given by

$$u = \frac{1}{2}\left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right) \tag{1.36}$$

As a wave, this energy flows as described by the Poynting Vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{1.37}$$

The *Intensity* of the light is defined as the time average of $\mathbf{S}$,

$$I = |\langle \mathbf{S} \rangle| \equiv \frac{1}{2}|\mathbf{E} \times \mathbf{H}| = \frac{n}{2\mu c}|\mathbf{E}|^2 \tag{1.38}$$

which has dimensions of watt per square centimeter.[15] In Eq. (1.38) we used the fact that $\omega/|\mathbf{k}| = c/n$. In vacuum, using the fact that $\varepsilon_0\mu_0 = 1/c^2$, we can write

$$I = \frac{1}{2\mu_0 c}|\mathbf{E}|^2 = \frac{1}{2}\varepsilon_0 c|\mathbf{E}|^2 \tag{1.39}$$

$$\simeq \frac{|\mathbf{E}|^2}{240\pi} \tag{1.39a}$$

13) By far field, we mean that the phase fronts are planes. This can be achieved near the source with lenses.

14) The boundary conditions of the problem might dictate a different Bessel solution. For example, if the origin were excluded, Bessel functions of the second kind, which are singular at the origin, would have to be considered as well.

15) Technically, the SI unit is watt per square meter but in the United States, it is typically expressed as watt per square centimeter .

In this form, I has dimensions of watt per square centimeter (watt per square meter) when the dimensions of $\mathbf{E}$ are volt per centimeter (volt per meter). The *Power*, P, delivered is the integrated intensity over the exposed area,

$$I = \frac{P}{A} \tag{1.40}$$

where A is the area.[16] A related quantity, the *Brightness*, which is sometimes referred to as the *Radiance*, takes into account the solid angle, $\Delta\Omega$, through which the intensity is delivered and is given by

$$B = \frac{I}{\Delta\Omega} \tag{1.41}$$

which has dimensions watts per steradian per square centimeter. It is interesting to note that an unfocused laser delivering 1 mW of power at 780 nm is considerably brighter than a 100 W light bulb, 1.7×10^7 W/sr-cm^2 for a typical laser beam[17] with $w_0 = 1$ mm and $\Delta\Omega = 2 \times 10^{-7}$ sr compared with 0.6 W/sr-cm^2 for a light bulb at a distance of 1 m radiating into 4π. Thus, a laser is considered very bright, which can do real damage to an unprotected eye. Finally, laser light can be further characterized by its *spectral brightness*, the brightness per unit optical bandwidth,

$$SB = \frac{B}{\Delta\nu} \tag{1.42}$$

with units as watts per steradian per square centimeter hertz. The *brightness* and *spectral brightness* are often confused with each other as well as with the *Luminance*, a photometric quantity referring to a perceived brightness related more to how the eye responds.

1.5 Polarization

As mentioned earlier, EM waves are actually vector waves, because $\mathbf{E}$ and $\mathbf{B}$ point in specific directions. *Polarization* captures this feature, and is defined in terms of the direction of $\mathbf{E}$.[18] The most general case is elliptical polarization, which has two limiting cases, linear and circular polarization. These names are so chosen because they describe the geometric shapes $\mathbf{E}$ that sweeps out while looking at the light along (parallel or antiparallel to) $\mathbf{k}$. We have already discussed that $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$ form a right-handed Cartesian triad so polarization also specifies the direction of $\mathbf{B}$. We will take $\hat{k} \equiv \hat{z}$ and focus on light that is perfectly polarized in the discussion that follows.

In general $\mathbf{E}$ will have two orthogonal components,

$$\begin{aligned}\mathbf{E}_1 &= \hat{x}E_{01}e^{i(kz-\omega t+\varphi_1)} \\ &= \hat{x}E_{01}e^{i(\overline{\xi}+\varphi_1)}\end{aligned} \tag{1.43}$$

$$\begin{aligned}\mathbf{E}_2 &= \hat{y}E_{02}e^{i(kz-\omega t+\varphi_2)} \\ &= \hat{y}E_{02}e^{i(\overline{\xi}+\varphi_2)}\end{aligned} \tag{1.44}$$

We will first consider the case where E_{01}, E_{02}, φ_1, and φ_2 are all real and

16) The area of a laser beam is given by πw_0^2, where w_0 is the beam radius.

17) For a diffraction limited laser beam, $\Delta\Omega = \pi\theta_d^2$ where $\theta_d = \lambda/\pi w_0$.

18) Another reason for considering only $\mathbf{E}$ is the magnitude of $\mathbf{B}$ relative to $\mathbf{E}$ is down by a factor of c. Thus at low intensities, $< 10^{14}$ W/cm^2, $\mathbf{E}$ dominates the physics.

Tab. 1.5 Various electromagnetic field quantities.

Quantity	Name	SI Unit
$c = 2.99792458 \times 10^8$	Light vacuum speed[a]	m/s
$\mu_0 = 4\pi \times 10^{-7}$	Vacuum permeability[a]	T-m/A (kg-m/A^2-s^2)
$\varepsilon_0 = 8.854187817\ldots \times 10^{-7}$	Vacuum permittivity	F/m (A^2-sec^4/kg-m^3)
$\mathbf{E}$	Electric field[b]	V/m (kg-m/A-s^3)
$\mathbf{D}$	Electric displacement	C/m^2
$\mathbf{B}$	Magnetic induction[c]	T (kg/A-s^2)
$\mathbf{H}$	Magnetic field	A/m
ρ	Charge density[b]	C/m^3
$\mathbf{j}$	Current density[b]	A/m^2
P	Power	W (kg/m^2-s^3)
$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$	Poynting vector	W/m^2 (kg/m^4-s^3)
$I \equiv \langle \lvert\mathbf{S}\rvert \rangle$	Intensity[b]	W/cm^2

[a] All defined to be exact.
[b] In the US, the explicit length measures for these quantities are given in centimeters, for example, volt per centimeter, watt per square centimeter, and so on.
[c] Sometimes called the *magnetic-flux density*.

time independent. Taking the real part of these fields,

$$\mathbf{E}_1 = \hat{x} E_{01} \cos(\overline{\xi} + \varphi_1) \tag{1.45}$$

$$\mathbf{E}_2 = \hat{y} E_{02} \cos(\overline{\xi} + \varphi_2) \tag{1.46}$$

leads to an equation of a conic section

$$\left(\frac{|\mathbf{E}_1|}{E_{01}}\right)^2 + \left(\frac{|\mathbf{E}_2|}{E_{02}}\right)^2 - 2\left(\frac{|\mathbf{E}_1|}{E_{01}}\right)\left(\frac{|\mathbf{E}_2|}{E_{02}}\right)\cos\varphi = \sin^2\varphi \tag{1.47}$$

and $\varphi = \varphi_2 - \varphi_1$. Equation (1.47) describes an ellipse when

$$\frac{\sin^2\varphi}{E_{01}^2 E_{02}^2} \geq 0 \tag{1.48}$$

Because the numerator and denominator are positive definite, Eq. (1.48) is always true.

Special Linear Case # 1: $\varphi = 0$ or $\varphi = \pi$
Equation (1.47) reduces to

$$\frac{|\mathbf{E}_1|}{E_{01}} = \frac{|\mathbf{E}_2|}{E_{02}} \tag{1.49}$$

which describes a straight line. Because the two component oscillate in phase, this case leads to linear polarization. The case for $\varphi = 0$ and $\varphi = \pi$ are orthogonal to each other.

Special Circular Case # 2: $\varphi = \pm\pi/2$ and $E_{01} = E_{02}$
The equation reduces to

$$|\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 = E_{01}^2 \tag{1.50}$$

the equation of a circle of radius E_{01}. The two components are out of phase by half a wavelength (or period) but the magnitude of the resultant, E_{01}, is constant but sweeps out a circle leading to circular polarization. The sense of rotation depends on the sign of φ, with the minus (plus) sign producing light with positive (negative) helicity, where positive helicity obeys the right-hand rule, so if you look in the direction of propagation, the E-field rotates clockwise.[19)]

Special Elliptical Case # 3: $\varphi = \pm\pi/2$ and $E_{01} \neq E_{02}$
Equation (1.47) reduces to

$$\left(\frac{|\mathbf{E}_1|}{E_{01}}\right)^2 + \left(\frac{|\mathbf{E}_2|}{E_{02}}\right)^2 = 1 \tag{1.51}$$

which is an ellipse with the major axis aligned with the horizontal (vertical) axis when $E_{01} > E_{02}$ ($E_{01} < E_{02}$). The sense of rotation is the same as in special case # 2.

General Elliptical Case $E_{01} \neq E_{02}$
In the general elliptical polarization case, one has a rotated ellipse where the angle, α, of the major axis away from the $\hat{E}_1$ direction is given by

$$\tan 2\alpha = \frac{2E_{01}E_{02}}{E_{01}^2 - E_{02}^2}\cos\varphi \tag{1.52}$$

Note, when $E_{01} = E_{02}$, Eq. (1.52) cannot be used and one must go back to Eq. (1.47) to determine α.

1.5.1 Polarization Bookkeeping

There are several approaches to keeping track of the polarization of light, which is particularly important when light interacts with media that can either decrease the intensity or delay the transit time of one polarization or helicity relative to the other. Here, we mention two matrices, namely, the Jones and Mueller matrices. A more in depth discussion can be found in Ref. Goldstein, 2003. The Jones approach involves a set of 2×2 matrices with complex elements that transform two-element vectors that describe the complex amplitude and phase of the light. Every Jones vector corresponds to a physically realizable polarization configuration. The Mueller approach uses a set of 4×4 matrices with real elements, which have values of either 0 or ±1, to transform the Stokes vectors (see Section 1.5.3). However, some matrices do not represent real configurations.

1.5.2 Jones Matrices

In the Jones calculus Clark Jones, 1941, an initial complex field $\mathbf{E}$ is transformed to the final complex field $\mathbf{E}'$ via matrix multiplication

$$\begin{pmatrix} E'_x e^{i\phi'_x} \\ E'_y e^{i\phi'_y} \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \cdot \begin{pmatrix} E_x e^{i\phi_x} \\ E_y e^{i\phi_y} \end{pmatrix} \tag{1.53}$$

where the components of the matrix $\mathbf{J}$ for common elements are given in Table 1.6.

1.5.3 Mueller Matrices

The general Muller calculus is also a matrix operation,

$$\mathbf{S}' = \mathbf{M} \cdot \mathbf{S} \tag{1.54}$$

where $\mathbf{M}$ is a 4×4 matrix and $\mathbf{S}$ is the four-element Stokes vector, credited to Sir George Gabriel Stokes for their invention. Given the electric fields in Eqs (1.45) and (1.46), the four

19) It should be noted that some references define circular polarization in terms of right-hand and left-hand circular polarization. This definition traditionally corresponds to looking antiparallel to $\mathbf{k}$ so $\varphi = -\pi/2$ would lead to left-hand circular polarization.

Tab. 1.6 Jones matrices for common optical elements.

Optical element	Jones matrix
Linear polarizer $\parallel\hat{x}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear polarizer $\parallel\hat{y}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear polarizer at $\pm 45^\circ$	$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$
$\frac{1}{4}$-Wave plate, Fast axis $\parallel \begin{matrix}\hat{x} & (+) \\ \hat{y} & (-)\end{matrix}$	$e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}$
Circular polarizer, $\pm$ Helicity	$e^{i\pi/4}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$

Tab. 1.7 Mueller matrices for common optical elements.

Optical element	Mueller matrix
Linear polarizer $\parallel \begin{matrix}\hat{x} & (+) \\ \hat{y} & (-)\end{matrix}$	$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Linear polarizer at $\pm 45^\circ$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\frac{1}{4}$-Wave plate, Fast axis $\parallel \begin{matrix}\hat{x} & (+) \\ \hat{y} & (-)\end{matrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \pm 1 & 0 \end{pmatrix}$
Circular polarizer, $\pm$ Helicity	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}$

components of **S** are defined as

$$S_0 = |E_{01}|^2 + |E_{02}|^2 \tag{1.55}$$

$$S_1 = |E_{01}|^2 - |E_{02}|^2 \tag{1.56}$$

$$S_2 = |2E_{01}E_{02}\cos\varphi| \tag{1.57}$$

$$S_3 = |2E_{01}E_{02}\sin\varphi| \tag{1.58}$$

The Mueller matrices are given in Table 1.7.

1.6 Longitudinal Field Component

Another manifestation of the vector nature of light is that it has a longitudinal component, even though it is customary to ignore it. It must be noted that a pure plane wave exists over all space and has no transverse variation. The finite extent of the field turns out to be acceptable in many cases but one runs into problems with very intense light, particularly when focused. Lax, Louisell, and McKnight (1975) showed that a purely transverse field is not an exact solution to Maxwell's equations, but rather it is the zeroth-order solution to the paraxial approximation to Maxwell's equations. The exact solutions require a longitudinal component given by

$$E_z(x, y, z) = \frac{1}{ik}\mathbf{\nabla}_\perp \cdot \mathbf{E}_\perp \tag{1.59}$$

where $k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$ and the $\perp$ symbol indicate the transverse components of the $\mathbf{\nabla}$ operator and **E** field. Using Fourier analysis of the fields, Scully and Zubairy (1991) showed a field obeying the paraxial approximation,

$$k_x, k_y \ll k \tag{1.60}$$

$$k_z \simeq k\left(1 - \frac{1}{2}\frac{k_x^2 + k_y^2}{k^2}\right) \tag{1.61}$$

implies Eq. (1.59).

1.7 Diffraction

When light passes a sharp edge, it does not produce a sharp shadow. Also, when it passes through a circular hole, it does not produce a disk of the same size. Under the right conditions, it produces not only a larger spot but also rings. Furthermore, the transverse size of a laser beam expands as it propagates. These observations are elegantly described by diffraction theory. Diffraction falls into two classes – Fraunhofer and Fresnel. Fraunhofer diffractions describes what happens when the phase fronts are near plane waves, where the curvature of the field can be ignored. Fresnel diffraction takes curvature into account.

Huygens, in the late seventeenth century, suggested a description for wave propagation as a collection of individual spherical sources called *secondary sources*, the sum of which would make up the wavefront. It is a straightforward exercise to convince oneself that Huygens's principle can be used to construct a plane as well as other simple geometries. When applied to a hole, Huygens's approach leads to an emerging spherical wave, because part of the plane wave is blocked. This would appear to account for the observed spread. However, there is a difficulty. If the secondary waves are spherical, then there should also be part of the wave going backward. Huygens had to ignore this part of the wave. It turns that when considered more mathematically, this problem is corrected by what is call the *obliquity factor*.

The mathematical statement of the principle for a wave propagating in free space is the Fresnel-Kirchhoff integral formula,[20]

$$\psi_P = -\frac{ik}{4\pi}\psi_0 \iint \frac{e^{ik(r+\bar{r})}}{r\bar{r}} \times [\cos\theta(\mathbf{n},\mathbf{r}) - \cos\theta(\mathbf{n},\bar{\mathbf{r}})]\,dA \tag{1.62}$$

where the integral is over the area of the aperture. The distances, r and $\bar{r}$, between the aperture and observation point and aperture and source, respectively, are defined in Figure 1.3 as is $\mathbf{n}$, the normal to the surface, pointing toward the source. The angles between the vectors and the normal are represented by $\theta(\mathbf{n},\mathbf{r})$ and $\theta(\mathbf{n},\bar{\mathbf{r}})$.

Let's consider an example of an aperture. In the Fraunhofer limit, s and p are effectively a long way from the aperture. In this case, we can take the surface of the aperture to be a spherical cap such that F is constant. Thus, $\bar{r}$ and $\mathbf{n}$ are antiparallel always and $\cos\theta(\mathbf{n},\bar{\mathbf{r}}) = -1$.[21] Equation (1.62) then reduces to

$$\psi_P = -\frac{ik}{4\pi}A\psi_0 \int \frac{e^{ik(r+\bar{r})}}{r\bar{r}} \times [\cos\theta(\mathbf{n},\mathbf{r}) + 1]\,dA \tag{1.63}$$

---

20) The Fresnel-Kirchhoff integral formula of Eq. (1.62) can be derived from Green's theorem (see, for example, Fowles, 1968) for two functions that are continuous, integrable and satisfy the wave equation,

$$\iint (V\nabla_\perp U - U\nabla_\perp V)\,dA = \iiint \left(V\nabla^2 U - U\nabla^2 V\right) dV$$

where the first integral is over any closed surface and the second is over the volume enclosed.

21) When $\bar{r}$ and r are much larger than the aperture size, a spherical surface is not much different from a flat surface.

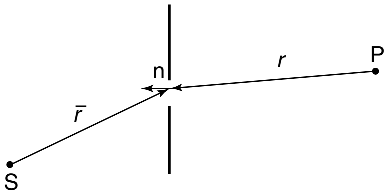

Fig. 1.3 Geometry for the Fresnel-Kirchhoff integral formula.

where $\cos\theta(\mathbf{n},\mathbf{r})+1$ is the obliquity factor mentioned above, which is zero for the wave that is going backwards, towards the source. Table 1.8 gives a few key results derived from Eq. (1.63).

From Table 1.8, we draw an important conclusion. The smaller the aperture, the larger the diffraction. In general, the diffraction angle is given by

$$\text{Slit} \longrightarrow \theta \sim \lambda/b \tag{1.64}$$

$$\text{Circle} \longrightarrow \theta \sim 1.22\lambda/2r \tag{1.65}$$

1.8 Interference

Interference is concerned with the superposition of waves. In general, the sum of wave solutions is also a solution to the wave equation. Here, we consider three examples: summing two waves of the same frequency, the general case of summing multiple waves of different frequencies and generation of short pulses of light by summing many frequencies with a precise phase relationship.

Tab. 1.8 Diffraction patterns from key apertures.*

Aperture	Intensity
Slit, width b	$I_0\,(\sin\beta/\beta)^2$
Rectangular slit, area $a \times b$	$I_0\,(\sin\alpha/\alpha)^2\,(\sin\beta/\beta)^2$
Circular aperture, radius r	$I_0\,\left(2J_1(\rho)/\rho\right)^2$
N slits, h spacing	$N^2 I_0\,(\sin\beta/\beta)^2\,(\sin N\gamma/N_{\sin}\gamma)^2$

*$\alpha = (\pi a/\lambda)\sin\phi$
$\beta = (\pi b/\lambda)\sin\theta$
$\gamma = (\pi h/\lambda)\sin\theta$
$\rho = (2\pi r/\lambda)\sin\theta$
ϕ diffraction angle, y direction.
θ diffraction angle, x direction.

1.8.1 Superposition: Single Frequency

This is straightforward to see with plane waves. Consider first two plane waves of the same frequency but with different real amplitudes and a relative phase between them,

$$A_T(\mathbf{r},t,\varphi_o) = a_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\varphi_1)} + a_2 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\varphi_2)} \tag{1.66}$$

The sum produces a new sinusoidal wave,

$$A_T(\mathbf{r},t,\varphi) = A_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\varphi_0)} \tag{1.67}$$

where

$$A_0 e^{i\varphi_0)} = a_1 e^{i\varphi_1} + a_2 e^{i\varphi_0} \tag{1.68}$$

We can determine A_0 and φ_0 in terms of $a_{1,2}$ and $\varphi_{1,2}$ by expanding the exponentials such that

$$A_0(\cos\varphi_0 + i\sin\varphi_0) = a_1(\cos\varphi_1 + i\sin\varphi_1) + a_2(\cos\varphi_2 + i\sin\varphi_3) \tag{1.69}$$

Equating the cosine (sine) terms on the left with those on the right and then dividing the sine terms by the cosine terms leads to

$$\tan\varphi_0 = \frac{a_1 \sin\varphi_1 + a_2 \sin\varphi_2}{a_1 \cos\varphi_1 + a_2 \cos\varphi_2} \quad (1.70)$$

At the same time, taking the modulus squared of Eq. (1.68) produces

$$\begin{aligned} |A_0|^2 &= |a_1|^2 + |a_2|^2 \\ &\quad + \left(a_1 a_2^* e^{i\Delta\varphi_0} + c.c.\right) \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos\Delta\varphi_0 \end{aligned} \quad (1.71)$$

where $\Delta\varphi_0 = \varphi_1 - \varphi_2$. Given $a_{1,2}$ and $\varphi_{1,2}$, A_0 (the intensity) and φ_0 can be found from Eqs (1.70) and (1.71). Equation (1.71) is known as the *coherent sum of the two waves*. That is, one adds the amplitudes before squaring to get the total intensity. The intensity is proportional to the square of the amplitude so it is also possible to write Eq. (1.71) as

$$I_T = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\Delta\varphi_0 \quad (1.72)$$

The third term in (Eqs 1.71 and 1.72) is sometimes called the *interference term* and plays an important role in describing the intensity of the resultant wave. Consider the case where $a_1 = a_2$ so $I_1 = I_2 = I$. When $\Delta\varphi_0 = 2m\pi$ $(m = 0, 1, 2, \ldots)$, the two waves are said to be in phase, in which case

$$I_T = (a_1 + a_2)^2 = 4I \quad (1.73)$$

When $\Delta\varphi_0 = (2m + 1)\pi/2$, we have the opposite extreme,

$$I_T = (a_1 - a_2)^2 = 0 \quad (1.74)$$

When $a_1 \neq a_2$, the two extremes give resultants with maximum and minimum I_T respectively.

In the more general case of many waves, all with the same frequency, we have

$$\begin{aligned} A_T(\mathbf{r}, t, \varphi_0) &= A_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi_0)} \\ &= \sum_{j=1}^{N} a_j e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi_j)} \end{aligned} \quad (1.75)$$

where

$$I_T = |A_0|^2 = \sum_{j=1}^{N} |a_j|^2 + \frac{1}{2} \sum_{j\neq k}^{N} \left(a_j a_k^* e^{i(\varphi_j - \varphi_k)} + c.c.\right) \quad (1.76)$$

and

$$\tan\varphi_0 = \frac{\sum_{j=1}^{N} a_j \sin\varphi_j}{\sum_{j=1}^{N} a_j \cos\varphi_j} \quad (1.77)$$

Again, the resultant is a sinusoidal wave with an intensity given by a coherent sum. In the case where all the amplitudes are the same so that each wave has an intensity I,

$$I_T = N^2 I \quad (1.78)$$

In the case where $\Delta_{j,k} = \varphi_j - \varphi_k$ is not well defined but varies randomly with time, it is straightforward to show that the second sum in Eq. (1.76) vanishes by writing the exponentials in terms of sines and cosines and using the fact that the time average of $\sin\Delta_{j,k} \to 0$ as does that of $\cos\Delta_{j,k}$. Thus, the interference terms vanish. In the case where all amplitudes are the same, the resultant wave corresponds to an incoherent sum of the contributors,

$$I_T = NI \quad (1.79)$$

For an incoherent sum, one squares first and then adds the intensities.

1.8.1.1 Interferometry

An entire field of study with industrial applications is built upon an equation similar to Eq. (1.72). The most general situation is where a beam of light is divided into two with each traveling different paths and brought back together. Because the two beams came from the same source, and if the path length difference is not too large, so that the two beams are still in phase, the resultant intensity will be the same as Eq. (1.72) except that $\Delta\varphi_0 \to \delta$ in the argument of the interference term where

$$\delta = k\Delta l \tag{1.80}$$

with Δl being the path length difference between the two arms. In this case, constructive interference occurs when $\Delta l = n\lambda$, whereas destructive interference occurs when $\Delta l = (2n+1)\lambda/2$, where n is a positive integer. Two-beam interferometry exploits interference patterns to measure inhomogeneities and defects in material.

1.8.2 Superposition: Multiple Frequencies

Superposition involving waves of different frequencies leads to some very interesting possibilities such as ultrashort busts of light. The general principle of summing waves with different frequencies can be understood in the special case where the amplitude and phase are the same for each wave:

$$\begin{aligned} A_T(\mathbf{r}, t) &= A_0 \exp[i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)] \\ &\quad + A_0 \exp[i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)] \\ &= 2A_0 \exp\left[\frac{i}{2}(\Delta\mathbf{k} \cdot \mathbf{r} - \Delta\omega t)\right] \\ &\quad \times \exp\left[\frac{i}{2}(\mathbf{k}_m \cdot \mathbf{r} - \omega_m t)\right] \end{aligned} \tag{1.81}$$

where

$$\Delta\mathbf{k} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2) \tag{1.82}$$

$$\mathbf{k}_m = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) \tag{1.83}$$

$$\Delta\omega = \frac{1}{2}(\omega_1 - \omega_2) \tag{1.84}$$

$$\omega_m = \frac{1}{2}(\omega_1 + \omega_2) \tag{1.85}$$

Equation (1.81) represents a wave oscillating at the mean of the two frequencies, ω_m, and modulated by a temporal and spatial envelope given by $2A_0 \exp[\frac{i}{2}(\Delta\mathbf{k} \cdot \mathbf{r} - \Delta\omega t)]$. Figure 1.4 shows examples of adding two waves with different frequencies. Unlike the case of equal frequencies, in this case, the two sinusoidal waves produce a wave that is periodic but not sinusoidal. Such waves are called *anharmonic*. For the sum of two waves, we have two different speeds. As with a single frequency, we again have a phase velocity – the ratio between the average frequency and wavenumber, $v_{ph} = \omega_m/|\mathbf{k}|$. But, we have a new speed that goes by the name of the group velocity, the speed with which the envelope moves, $v_g = \Delta\omega/\Delta|\mathbf{k}|$.

When a wave is composed of many frequencies, $\omega \to \omega(k)$. Typically, the frequencies are grouped around a central frequency, $\omega(k_0)$, allowing $\omega(k)$ to be expanded into a Taylor series,

$$\omega(k) = \omega(k_0) + (k - k_0)\left.\frac{d\omega}{dk}\right|_{k_0} + \cdots \tag{1.86}$$

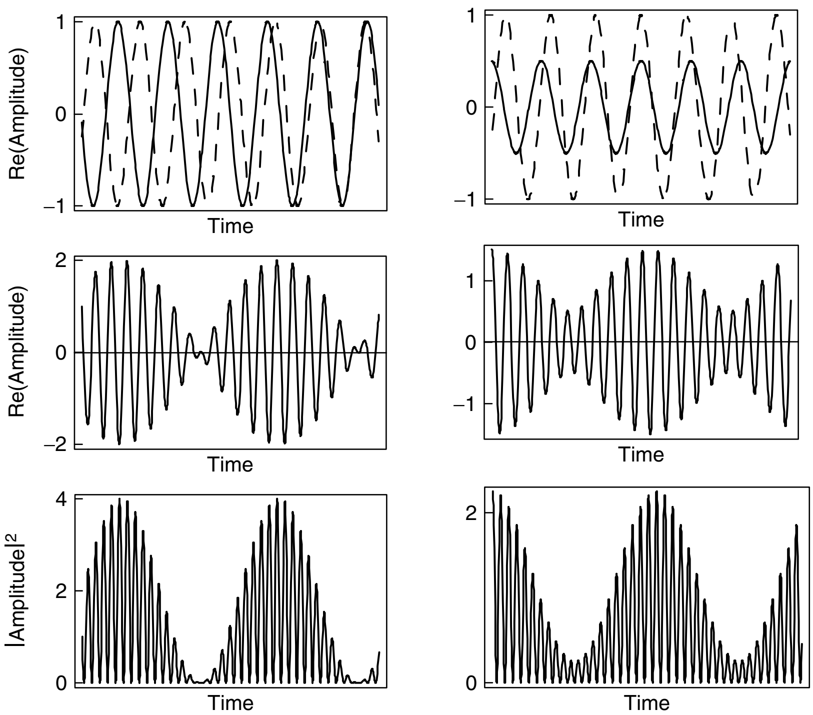

Fig. 1.4 Superposition of two waves, $A_1 \exp[i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t) + \varphi_1] + A_2 \exp[i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t) + \varphi_2]$ where the top row corresponds to the individual amplitudes (A_1 solid curves), the middle row to the resultant sum of amplitudes and the bottom row to the square of the modulus of the resultant amplitudes: (left column) $A_1 = A_2$, $\Delta\varphi = \pi/2$ and $\omega_1/\omega_2 = 0.9$; (right column) $A_1 = A_2/2$, $\Delta\varphi = 0$ and $\omega_1/\omega_2 = 0.9$, where $\Delta\varphi = \varphi_1 - \varphi_2$.

In this case, the addition of the various frequency components is more easily handled via Fourier analysis (see, for example Arfken and Weber (2000) and Boas (2006) for a review), where the amplitude $A(t)$ ($A(z)$) in the time (coordinate space) domain is linked to $\tilde{A}(\omega)$ ($\tilde{A}(k)$) in the frequency (spatial frequency) domain through

$$A(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(\omega) e^{-i\omega t} d\omega \quad (1.87)$$

$$\tilde{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(t) e^{i\omega t} dt \quad (1.88)$$

for time-frequency and

$$A(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(\mathbf{k}) e^{-ikz} dk \quad (1.89)$$

$$\tilde{A}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(z) e^{ikz} dz \quad (1.90)$$

for coordinate space - frequency space. Thus, in Fourier components, the electric field for a scalar wave propagating in the z-direction can be written as

$$E(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}(k) e^{-i[kz + \omega(k)t]} dk \quad (1.91)$$

In many cases, $d\omega/dk$ is the appropriate and more general expression for the group velocity. This can be seen by substituting the first two terms of Eq. (1.86) into Eq. (1.91),

$$E(z, t) = \frac{1}{\sqrt{2\pi}} e^{i[k_0(d\omega/dk)|_{k_0} - \omega(k_0)]t} \times \int_{-\infty}^{\infty} \tilde{E}(k) e^{-i[z+(d\omega/dk)|_{k_0} t]k} dk \quad (1.92)$$

However, Eq. (1.90) implies

$$\tilde{E}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(z, t=0) e^{ikz} dz \quad (1.93)$$

which allows Eq. (1.92) to be written as

$$E(z, t) = \frac{e^{i[k_0(d\omega/dk)|_{k_0} - \omega(k_0)]t}}{2\pi} \times \int_{-\infty}^{\infty} E(z', 0) dz' \times \int_{-\infty}^{\infty} e^{i(z'-z-(d\omega/dk)|_{k_0} t)k} dk \quad (1.94)$$

where we do the k integration first. The last integral is just $\delta\left(z' - z - \frac{d\omega}{dk}\big|_{k_0} t\right)$, from which we get

$$E(z, t) = \frac{1}{2\pi} E(z + d\omega/dk|_{k_0} t, 0) \times e^{-i[\omega(k_0) - k_0(d\omega/dk)|_{k_0}]t} \quad (1.95)$$

By inspection, it is clear that the envelope in Eq. (1.95) moves with speed $d\omega/dk|_{k_0}$ and the carrier oscillates with frequency $\omega(k_0) - k_0 \frac{d\omega}{dk}|_{k_0}$ under the envelope. Thus, we define the group velocity as

$$v_g = \frac{d\omega}{d|\mathbf{k}|} \quad (1.96)$$

In vacuum $v_{ph} = v_g$. However, if the medium through which the wave propagates is dispersive, $n \rightarrow n(\lambda)$ so that $dn/d|\mathbf{k}| \neq 0$, the two velocities can be very different. Thus, it is often convenient to write v_g in a form that includes the dispersion explicitly,

$$v_g = \frac{c}{n} \left(1 - \frac{|\mathbf{k}|}{n} \frac{dn}{d|\mathbf{k}|}\right) \quad (1.97)$$

The group velocity is typically the speed with which information is transmitted. It is important to remember that the group velocity is actually only the first term in a series and in cases where $dn/d|\mathbf{k}|$ changes very rapidly or is anomalous (i.e., negative), higher order terms must be kept to determine the speed with which information travels correctly.

1.8.3 Short Pulses

Figure 1.4 shows the basic idea for generating pulses of light of short duration. Specifically, in this case, two frequencies with well-defined relative phase (i.e., fixed in time) are summed in the frequency domain to provide a new wave with beats in the time domain. As additional frequencies are added, the temporal width of the beat envelope narrows. To gain a better understanding of the relationship between the length of the pulse train and its *bandwidth* or number of frequencies required to sum in order to produce it, we will turn the problem around and start with an idealized pulse train in the time domain. Figure 1.5, for example, shows two finite length, idealized, pulse trains, one with three cycles and the other with six cycles.

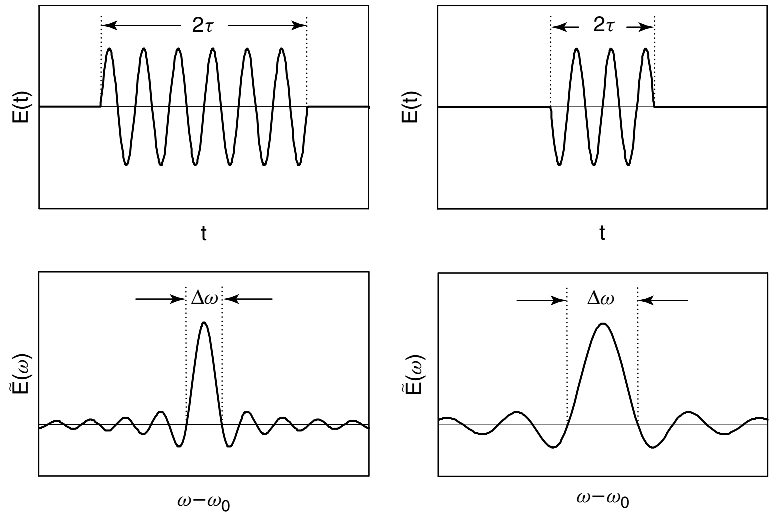

Fig. 1.5 Idealized short pulses formed by finite unit amplitude N-cycle pulse trains (top) with $N = 6$ (left) and $N = 3$ (right). Their respective Fourier transforms appear below with peak amplitudes of $\sqrt{\pi/2}N/\omega_0$ and the first zeros occurring at $\omega = \omega_0(1 \pm 1/N)$.

Mathematically, these obey

$$E(t) = \begin{cases} E_0 \sin \omega_0 t & \text{for } -\tau \le t \le \tau \\ 0 & \text{at other times.} \end{cases} \tag{1.98}$$

The length of this pulse is 2τ, where $\tau = N\pi/\omega_0$ with N being the number of cycles in the train. Using a Fourier analysis similar to that described above, the frequency spectrum is given by

$$\tilde{E}(\omega) = \frac{E_0}{\sqrt{2\pi}}\left[\frac{\sin \tau(\omega - \omega_0)}{\omega - \omega_0} - \frac{\sin \tau(\omega + \omega_0)}{\omega + \omega_0}\right] \tag{1.99}$$

At optical or near IR frequencies, because the second term is much smaller than the first, we can apply the Fourier transform to just the first term, which is also plotted in Figure 1.5. Clearly, the number of frequencies involved in the shorter pulse is larger than the number needed for the longer pulse. This inverse relationship between the length of the pulse in the time domain and the spread in the frequency domain is conveniently captured in the time-bandwidth product, $\tau\Delta\nu$. From Figure 1.5 and Eqs (1.98) and (1.99), it is clear that $\tilde{E}(\omega) = 0$ when $N\pi(\omega - \omega_0)/\omega_0 = \pm\pi$. Thus, $\Delta\omega = \omega_+ - \omega_- = 2\omega_0/N = 2\pi/\tau$, where $\omega_\pm = \omega_0 (1 \pm 1/N)$, which leads to

$$\tau\Delta\nu = 1 \tag{1.100}$$

It is interesting to note that if we multiply Eq. (1.100) by h, this time-bandwidth product satisfies the Heisenberg uncertainty principle,

$$\Delta t \Delta E \ge \frac{\hbar}{2} \tag{1.101}$$

where $\Delta E = h\Delta\nu$, $\hbar = h/2\pi$ and we substituted Δt for τ. The minimum is reached when the so-called minimum uncertainty wavepacket is prepared.[22)]

Ultrashort pulses are achieved by ''locking'' the frequency components that extends over a wide frequency range. The minimum width achievable by this technique corresponds to one complete cycle of light. At 800 nm, near the peak of the Ti : Sapphire laser, this is ~ 2.7 femtoseconds. For a more complete discussion on mode locking and the generation of ultrashort pulses, the reader is directed to the classic text by Siegman Siegman, 1986

1.9 Photons and Particles

We conclude by discussing the analogy between light and particle waves a bit further. Equations (1.13), (1.14) and (1.31) can be used to motive the time-independent Schrödinger wave equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = \mathcal{E}\psi \qquad (1.102)$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}(\mathcal{E} - V)\psi = 0 \qquad (1.103)$$

where $\hbar = h/2\pi$. Because λ for the particle is h/p, in free space, we postulate that k $(= 2\pi n/\lambda)$ in Eq. (1.31) must be proportional to p. Thus, in the absence of a potential (when $n = 1$)

$$k_{n=1}^2 = \frac{p^2}{h^2/4\pi^2} \qquad (1.104)$$

But,

$$p^2 = 2m\mathcal{E} \qquad (1.105)$$

22) The minimum spread criterion applies to conjugate variables such as time frequency and position momentum.

so

$$k_{n=1}^2 = \frac{2m}{\hbar^2}\mathcal{E}, \quad \text{which lead to Eq. 1.102.} \qquad (1.106)$$

To account for the potential we let $n^2 = (\mathcal{E} - V)/\mathcal{E} \neq 1$ so that $k^2 = n^2 k_{n=1}^2$ is just the coefficient of the second term in Eq 1.103.

Space does not permit a more in depth discussion of the quantum nature of light. The interested reader is directed to a recent review Smith and Raymer, 2007 and references therein.

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