CHAPTER 1

LAGRANGIAN AND HAMILTONIAN FORMULATIONS

1 INTRODUCTION

In this chapter we outline several approaches to treating problems of dynamics in classical mechanics, i.e., problems concerning motion in which forces are involved. We examine the force and the energy approaches, then discuss the Lagrangian and Hamiltonian formulations, and finally point out some variational approaches. Explicit expressions are given for the Lagrangians, Hamiltonians, and canonical momenta of various commonly encountered systems.

2 NEWTON'S LAW APPROACH

The force approach to non-relativistic dynamics makes use of Newton's second law which states that the force $\mathbf{F}$ on a body equals its rate of change of momentum $\dot{\mathbf{p}}$

$$\mathbf{F} = \dot{\mathbf{p}} = \frac{d\mathbf{p}}{dt} \tag{1}$$

Any reference frame in which this law holds is called an inertial frame. In one dimension the momentum for a particle of mass m and velocity v is $p = mv$, and Eq. (1) becomes

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} \tag{2}$$

where x is the position coordinate and a is the acceleration. The more general expression (1) must be used when the mass changes, as in nuclear decay problems involving the creation and annihilation of particles.

In dynamics we encounter several types of forces, such as:

$F = mg$	gravity near the earth's surface	(3a)
$\mathbf{F} = Gmm'\hat{\mathbf{r}}/r^2$	Newton's law of gravity	(3b)
$\mathbf{F} = qq'\hat{\mathbf{r}}/4\pi\varepsilon_0 r^2$	Coulomb's law	(3c)
$F = -k(x - x_0)$	harmonic restoring force from the equilibrium position x_0	(3d)
$F = -\mu N$	static or kinetic friction	(3e)
$F = -k\lvert v\rvert^n$	frictional force, often $n = 1$ as in Stokes' law $F = 6\pi\eta rv$ for streamline flow, or $n = 2$ with turbulence	(3f)
$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	Lorentz force on the charge q	(3g)

Ordinarily it is easier to solve non-relativistic elementary mechanics problems by balancing the energy at the beginning (b) and at the end (e) of the interaction. For a many-particle system in which each particle i has the kinetic energy $\frac{1}{2}m_i v_i^2$ and the potential energy V_i, we can write

$$\sum \tfrac{1}{2}m_i v_{ib}^2 + \sum V_{ib} = \sum \tfrac{1}{2}m_i v_{ie}^2 + \sum V_{ie} \tag{4}$$

where the summations are over the particles of the system. If the particles interact with each other, then we can add double summations over the initial and final interaction energies V_{ij}^b and V_{ij}^e to this equation. This energy approach is valid for conservative systems for which there is no dissipation.

3 LAGRANGIAN FORMULATION

In the Langrangian approach the kinetic and potential energies are expressed in terms of generalized coordinates q_i and the velocities $\dot{q}_i$ of each particle i. The Lagrangian itself, $L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_n, t)$, is the difference between the total kinetic energy T and the total potential energy V of the system

$$L = T - V \tag{5}$$

Examples of some Lagrangians are:

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}k(x - x_0)^2 \qquad \text{harmonic oscillator} \tag{6a}$$

$$\left.\begin{aligned} L &= -mc^2/\gamma - q\phi + q\mathbf{A}\cdot\mathbf{v} \\ L &\approx \tfrac{1}{2}mv^2 - q\phi + q\mathbf{A}\cdot\mathbf{v} - mc^2 \end{aligned}\right\} \quad \text{charge } q \text{ in electromagnetic fields } [\gamma = (1-\beta^2)^{-1/2}] \tag{6b}$$

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + k/r \qquad \text{Kepler problem of the earth in orbit around the sun} \tag{6c}$$

$$L = \tfrac{1}{2}I_\perp(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I_\parallel(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta \tag{6d}$$

Symmetric top with one point fixed using Euler angles
ψ = rotation about top axis
ϕ = precession about vertical
θ = angle of inclination from vertical

In writing Lagrangians it is good to keep in mind the following details about three commonly used coordinate systems with their respective differential lengths $ds_i = h_i dq_i$.

$$\text{cartesian:} \quad \begin{matrix} x, y, z \\ dx, dy, dz \end{matrix} \tag{7a}$$

$$\text{cylindrical:} \quad \begin{matrix} \rho, \phi, \mathrm{z} \\ d\rho, \rho d\phi, dz \end{matrix} \tag{7b}$$

$$\text{spherical:} \quad \begin{matrix} r, \theta, \phi \\ dr, rd\theta, r\sin\theta d\phi \end{matrix} \tag{7c}$$

Lagrangians are important because, by Hamilton's principle, the motion from time t_1 to time t_2 follows a path that makes the line integral

$$I = \int_{t_1}^{t_2} L dt \tag{8}$$

a stationary value. Using this principle we can show that Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, \ldots, N \tag{9}$$

is satisfied for each of the N coordinates q_i and their velocities $\dot{q}_i$.

The motion is often limited by constraints. Of particular interest is a holonomic constraint, which can be expressed in terms of equations involving the positions $\mathbf{r}_i$ of the particles, but not the velocities

$$f(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0 \tag{10}$$

If the constraint contains an inequality rather than an equal sign, such as the condition $r^2 - a^2 \geq 0$ for being outside a sphere of radius a, then it is not holonomic. The differential df of the function $f(\mathbf{q}_i, t)$

$$df = \sum \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial t} dt = 0 \tag{11}$$

where the summation is over the N particles, may be written in the form

$$\sum a_i dq_i + a_t dt = 0 \tag{12}$$

where $a_i = \partial f / \partial q_i$. If there is more than one constraint equation, then we add an index, writing a_{ji} and a_{jt}. The coefficients a_{ji} of the n constraint equations written as first-order differential equations

$$\sum a_{ji}\dot{q}_i + a_{jt} = 0 \quad j = 1, 2, \ldots, n \tag{13}$$

enter the N Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum a_{ji}\lambda_j \quad i = 1, 2, \ldots, N \tag{14}$$

through the n Lagrange multipliers λ_j, where the summation over j is from 1 to n. These are now $N + n$ equations to solve and n unknown Lagrange multipliers to be determined. Each λ_j usually has the dimension of force or torque, and in a typical case it is a reaction force.

Sometimes the constraint can be written in the form of Eq. (12) or (13) even though no function $f(q, t)$ exists. Such a constraint might be called pseudoholonomic; it is not holonomic but it is adequate for the application of the Lagrange multiplier method (14).

Sometimes friction can be taken into account with the aid of a velocity dependent Rayleigh dissipation function D

$$D = \tfrac{1}{2}\sum (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2) \tag{15}$$

associated with the frictional force $\mathbf{F}_f$

$$\mathbf{F}_f = -\nabla_v D \tag{16}$$

where the gradient is with respect to the velocities. The associated Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0 \quad i = 1, 2, \ldots, N \tag{17}$$

The Rayleigh dissipation function corresponds to the force of Eq. (3f) with $n = 1$.

Each coordinate q_j has a conjugate momentum p_j defined by the expression

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{18}$$

Sometimes the conjugate momentum is a linear momentum, as $m\dot{x}$, or an angular momentum, as $I\dot{\theta}$, where I is the moment of inertia, but it can also be more complicated than that, as in the case of a charge q moving in the presence of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Several conjugate momenta are:

$$p_i = \gamma m v_i + qA_i \qquad \text{charge } q \text{ in an electromagnetic field} \tag{19a}$$

$$\left.\begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \end{aligned}\right\} \qquad \text{Kepler problem} \tag{19b}$$

$$\left.\begin{aligned} p_\theta &= I_\perp \dot{\theta} \\ p_\psi &= I_\parallel(\dot{\psi} + \dot{\phi}\cos\theta) \\ p_\phi &= (I_\parallel \cos^2\theta + I_\perp \sin^2\theta)\dot{\phi} + I_\parallel \dot{\psi}\cos\theta \end{aligned}\right\} \text{symmetric top with one one point fixed in Euler angles} \tag{19c}$$

4 HAMILTONIAN FORMULATION

We have discussed the properties of the Lagrangian $L(q_i, \dot{q}_i, t)$, which is a function of the generalized coordinates and velocities. There is another function, called the Hamiltonian $\mathcal{H}(q_i, p_i, t)$, which depends on the coordinates q_i and their associated conjugate momenta p_i defined by Eq. (18), and it is formed from the Lagrangian through the following Legendre transformation

$$\mathcal{H}(q_i, p_i, t) = \sum \dot{q}_i p_i - L(q_i, \dot{q}_i, t) \tag{20}$$

where the summation is over the coordinates. To carry out this transformation each conjugate momentum p_i is determined by differentiating the Lagrangian using Eq. (18). Then the resulting equations are solved for the generalized velocities $\dot{q}_i$ and the latter are put into Eq. (20). This provides the Hamiltonian $\mathcal{H}$ expressed as a function of the q_i, p_i and t variables.

Sometimes this procedure is easy, as with the angle of inclination θ of the symmetric top (19c) for which we have

$$\dot{\theta} = p_\theta / I_\parallel \tag{21}$$

It is more complicated to determine the other two symmetric top angles ψ and ϕ which require the solution of two simultaneous equations.

The equations of motion are found from the canonical equations of Hamilton

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \tag{22a}$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{22b}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t} \tag{22c}$$

These $2N$ first-order Hamilton differential equations replace the N second-order Lagrange equations (8).

Examples of some Hamiltonians are:

$$\mathcal{H} = \frac{p^2}{2m} + \tfrac{1}{2}k(x - x_0)^2 \qquad \text{harmonic oscillator} \tag{23a}$$

$$\left.\begin{aligned} \mathcal{H} &= [(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4]^{1/2} + q\phi \\ &\approx \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi + mc^2 \end{aligned}\right\} \qquad \begin{array}{l}\text{charge } q \text{ in an} \\ \text{electromagnetic field}\end{array} \tag{23b}$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \qquad \text{Kepler problem} \tag{23c}$$

$$\mathcal{H} = \frac{p_\theta^2}{2I_\perp} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_\perp \sin^2\theta} + \mathrm{M}g\ell \cos\theta \qquad \text{symmetric top} \tag{23d}$$

Coordinates that do not appear explicitly in the Lagrangian and Hamiltonian are called cyclic or ignorable. We see from Eqs. (22b) that the momentum p_i conjugate to such a coordinate is a constant of the motion, independent of the time, which we will denote by α_i

$$\dot{p}_i = 0 \tag{24}$$

$$p_i = \alpha_i \tag{25}$$

Because of this it is ordinarily easier to use the Hamiltonian formulation for writing the equations of motion for cyclic coordinates, and the Lagrangian approach for the non-cyclic coordinates.

When some coordinates $q_i, \ldots, q_c$ are present explicitly in the Lagrangian and the remainder $q_{c+1}, \ldots, q_N$ are cyclic, then it is convenient to restrict the term $\sum q_i p_i$ of Eq. (20) to a summation over the cyclic coordinates and thereby form the Routhian R

$$R = R(q_1, \ldots, q_N, \dot{q}_i, \ldots, \dot{q}_c, p_{c+1}, \ldots, p_N) \tag{26}$$

which is a function of the non-cyclic velocities $\dot{q}_i$ and the constant momenta p_i. This permits Hamilton's equations to be written for the cyclic coordinates and Lagrange's equations for the non-cyclic ones. For example, the Kepler problem has the cyclic coordinate θ and hence from Eqs. (6c) and (19b) the Routhian $R(r, \theta, \dot{r}, p_\theta)$ is given by

$$R(r, \theta, \dot{r}, p_\theta) = \tfrac{1}{2}m\dot{r}^2 + p_\theta^2/2mr^2 - k/r \tag{27}$$

where, since θ is cyclic, the angular momentum $p_\theta = mr^2\dot{\theta}$ is constant in time.

5 VARIATIONAL PRINCIPLES AND VIRTUAL DISPLACEMENTS

There are several general principles associated with the subject of mechanics. We have already encountered Hamilton's principle (8), which states that the line integral of the Lagrangian over time has a stationary value for the correct path of motion. This can be expressed differently by saying that the variation of the line integral I of Eq. (8) with fixed end points t_1 and t_2 is zero

$$\delta I = \delta \int_{t_1}^{t_2} L dt \tag{28}$$

In other words, as infinitesimal deviation in the path of integration about its stationary trajectory does not change the value of the integral.

Now let us consider infinitesimal displacements $\delta \mathbf{r}_i$ in the presence of an applied force $\mathbf{F}^{\text{app}}$ which is holding a system in equilibrium. The principle of virtual work states that the work done by the applied force, called the virtual work, is zero

$$\sum \mathbf{F}_i^{\text{app}} \cdot \delta \mathbf{r}_i = 0 \tag{29}$$

and this provides the condition for equilibrium in statics. In dynamics when the applied forces can cause accelerations $\ddot{x} = \dot{p}_x/m$, we have D'Alembert's principle

$$\sum (\mathbf{F}_i^{\text{app}} - \dot{p}_i) \cdot \delta r_i = 0 \tag{30}$$

which reduces to the principle of virtual work for static conditions.