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Harmonic Oscillators and the Universal Language of Science

Harmonic motion is one of the most frequently observed phenomena that occurs in nature. We watch oscillations of pendulums, we admire the waves on the tranquil water surface, hear the propagating sound in the air, and see light. All these quite different physical phenomena exhibit local or moving periodic motions of matter described by two trigonometric functions, sine and cosine. This mathematical similarity leads to fundamental concepts of a harmonic oscillator and its "moving brother" the plane wave – a harmonic oscillator in k-space. Both oscillators unify mathematically periodic motions in various physical systems and form a universal language of science [1–5].

Small oscillations of any physical system in the vicinity of equilibrium are harmonic ones as long as the Taylor expansion of the potential energy in this case begins with the quadratic terms [6]. The system of magneto-ordered spins is not an exception. Perturbations of a magnetic order can be regarded as an evolution of waves in a nonlinear medium excited to some definite level or as an evolution of a nonideal gas of quasi particles known as magnons. This approach explains many dynamic, thermodynamic and kinetic phenomena in magnets [7–13].

In this chapter we shall briefly present the basic properties of a harmonic oscillator, its direct connection with the rotation, and introduce variables convenient for the description of spin motion. For simplicity, here we shall use the classical description in a way that resembles the quantum mechanical approach. The goal is to refresh the reader's knowledge of harmonic oscillators and become familiar with the notation used in this book.

1.1
Harmonic Oscillator

The harmonic oscillator is described by the following equation for a generalized coordinate q:

$$\frac{d^2 q}{dt^2} + \omega^2 q = 0 \,. \tag{1.1}$$

Nonequilibrium Magnons, First Edition. Vladimir L. Safonov.

Here ω is the frequency of oscillations. The general solution of (1.1) is

$$q(t) = A_h \cos(\omega t - \phi) \ , \tag{1.2}$$

where A_h is the amplitude and ϕ is the phase defined by the initial conditions

$$\begin{aligned} q(0) &= A_h \cos\phi \ , \\ \frac{dq(0)}{dt} &= -A_h \omega \sin\phi \ . \end{aligned} \tag{1.3}$$

For example, we can obtain (1.1) with

$$\omega = \sqrt{\frac{\kappa}{m}} \tag{1.4}$$

for a mass m on a spring (Figure 1.1a) with the force constant κ using Hamilton's equations of motion

$$\begin{aligned} \frac{dx}{dt} &= \{x, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \ , \\ \frac{dp}{dt} &= \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial x} = -\kappa x \ , \end{aligned} \tag{1.5}$$

where

$$\{\mathcal{A}, \mathcal{B}\} \equiv \frac{\partial \mathcal{A}}{\partial x}\frac{\partial \mathcal{B}}{\partial p} - \frac{\partial \mathcal{B}}{\partial x}\frac{\partial \mathcal{A}}{\partial p} \tag{1.6}$$

are the Poisson brackets and the energy of the system, the Hamiltonian, is defined by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{\kappa x^2}{2} \ . \tag{1.7}$$

Here $q = x$ and $p = mx/dt$ denotes the momentum.

Problem 1.1. Write analogous equations for a charge Q in the LC circuit (Figure 1.1b) denoting $q = Q$, $p = LdQ/dt$, $\kappa = 1/C$ and $\omega = \sqrt{1/LC}$. The reader can continue a number of examples.

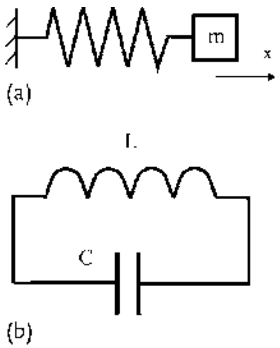

Figure 1.1 (a) Mass on a spring, and (b) LC circuit.

1.1.1 Complex Canonical Variables

Introducing the complex variables,

$$
\begin{aligned}
a^* &= \frac{1}{2}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right) ,\\
a &= \frac{1}{2}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) ,
\end{aligned}
\tag{1.8}
$$

where

$$
x_0 \equiv \sqrt{\frac{\hbar}{2m\omega}} , \quad p_0 \equiv \sqrt{\frac{\hbar m\omega}{2}} , \tag{1.9}
$$

we obtain classical analogs of creation and annihilation operators that are usually used in quantum mechanics to describe the harmonic motion of a mass on a spring. Here Planck's constant $\hbar$ is used as a dimensional constant in order to have dimensionless a^* and a. The Hamiltonian (1.7) acquires the form

$$
\mathcal{H} = \hbar\omega a^* a \tag{1.10}
$$

and the equations in (1.5) become

$$
\begin{aligned}
i\frac{da}{dt} &= \left[a, \frac{\mathcal{H}}{\hbar}\right]_c = \frac{\partial\mathcal{H}/\hbar}{\partial a^*} = \omega a ,\\
i\frac{da^*}{dt} &= \left[a^*, \frac{\mathcal{H}}{\hbar}\right]_c = -\frac{\partial\mathcal{H}/\hbar}{\partial a} = -\omega a^* .
\end{aligned}
\tag{1.11}
$$

Here the classical analog of the commutator is defined by

$$
[\mathcal{A}, \mathcal{B}]_c \equiv \frac{\partial\mathcal{A}}{\partial a}\frac{\partial\mathcal{B}}{\partial a^*} - \frac{\partial\mathcal{B}}{\partial a}\frac{\partial\mathcal{A}}{\partial a^*} . \tag{1.12}
$$

Note that the equations in (1.11) are another form of the principal equation (1.1) for a harmonic oscillator. As can be seen from the solution of (1.11)

$$
\begin{aligned}
a &= a(0)\exp(-i\omega t) ,\\
a^* &= a^*(0)\exp(i\omega t) ,
\end{aligned}
\tag{1.13}
$$

complex canonical variables describe rotation (Figure 1.2) and therefore can be convenient for the description of the magnetic moment precession.

Problem 1.2. Write out the complex canonical variables for the LC circuit.

Quantum mechanical properties of a harmonic oscillator are given in Appendix A.

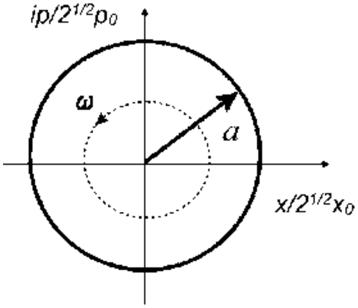

Figure 1.2 Rotation of complex canonical variables.

1.2
Classical Rotation

Hamilton's equations of motion for a particle in classical mechanics can be written as

$$\begin{aligned}\frac{dx_\ell}{dt} &= \{x_\ell, \mathcal{H}\}\,,\\ \frac{dp_\ell}{dt} &= \{p_\ell, \mathcal{H}\}\,,\end{aligned} \tag{1.14}$$

where $\mathcal{H} = \mathcal{H}(x_\ell, p_\ell)$ is the Hamiltonian (energy), $x_\ell = x, y, z$ are coordinates of the particle and $p_\ell = m dx_\ell/dt$ is their momentum (m is the mass).

$$\{\mathcal{A}, \mathcal{B}\} \equiv \sum_\ell \left(\frac{\partial \mathcal{A}}{\partial x_\ell}\frac{\partial \mathcal{B}}{\partial p_\ell} - \frac{\partial \mathcal{B}}{\partial x_\ell}\frac{\partial \mathcal{A}}{\partial p_\ell}\right) \tag{1.15}$$

is the Poisson bracket.

Equations (1.14) can be rewritten in the form which completely resembles quantum Heisenberg equations:

$$\begin{aligned}i\hbar\frac{dx_\ell}{dt} &= [x_\ell, \mathcal{H}]_c\,,\\ i\hbar\frac{dp_\ell}{dt} &= [p_\ell, \mathcal{H}]_c\,,\end{aligned} \tag{1.16}$$

where

$$[\mathcal{A}, \mathcal{B}]_c \equiv i\hbar\{\mathcal{A}, \mathcal{B}\} \tag{1.17}$$

is the classical analog of the commutator. Planck's constant $\hbar$ is introduced as a dimensional constant for simple correspondence with quantum equations. For example, the coordinate and momentum classical commutator corresponds to the quantum commutation rule:

$$[x_i, p_j]_c = i\hbar\delta_{ij}\,. \tag{1.18}$$

A rotation of particles in a rigid body is associated with their sum of angular momenta (e.g., [6]) $\boldsymbol{J} = \sum \boldsymbol{r} \times \boldsymbol{p}$ or, in the explicit form,

$$J^x = \sum y p_z - z p_y \,, \quad J^y = \sum z p_x - x p_z \,, \quad J^z = \sum x p_y - y p_x \,. \tag{1.19}$$

Taking into account (1.19) and (1.18), we can obtain

$$\begin{aligned}
[J^x, J^y]_c &= i\hbar J^z \,, \\
[J^y, J^z]_c &= i\hbar J^x \,, \\
[J^z, J^x]_c &= i\hbar J^y \,.
\end{aligned} \tag{1.20}$$

Hamilton's equations for the angular momentum can be written in the following form:

$$i\hbar \frac{d\boldsymbol{J}}{dt} = [\boldsymbol{J}, \mathcal{H}]_c \,. \tag{1.21}$$

So far as the absolute value $J = |\boldsymbol{J}|$ is conserved, we can introduce new canonical variables J^z and the angle ϕ between the projection of the angular momentum in the x–y plane and the x-axis:

$$\begin{aligned}
J^x &= \sqrt{J^2 - (J^z)^2} \cos\phi = \frac{\partial J^y}{\partial \phi} \,, \\
J^y &= \sqrt{J^2 - (J^z)^2} \sin\phi = -\frac{\partial J^x}{\partial \phi} \,.
\end{aligned} \tag{1.22}$$

It is easy to check that the Poisson bracket in this case becomes

$$\{\mathcal{A}, \mathcal{B}\} \equiv \frac{\partial \mathcal{A}}{\partial \phi} \frac{\partial \mathcal{B}}{\partial J^z} - \frac{\partial \mathcal{B}}{\partial \phi} \frac{\partial \mathcal{A}}{\partial J^z} \,. \tag{1.23}$$

If the energy depends on the J_z component only, there is a gyroscopic precession around the z-axis. The equations of this precession follow from (1.21) and (1.23):

$$\frac{dJ_x}{dt} = -J_y \frac{\partial \mathcal{H}}{\partial J_z} \,, \quad \frac{dJ_y}{dt} = J_x \frac{\partial \mathcal{H}}{\partial J_z} \,. \tag{1.24}$$

In vector form we have

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{J} \times \left(-\frac{\partial \mathcal{H}}{\partial \boldsymbol{J}} \right) \,. \tag{1.25}$$

1.2.1
Classical Spin and Magnetic Resonance

Now we shall consider a classical spin momentum $\hbar \boldsymbol{S}$ which has properties analogous to the angular momentum $\boldsymbol{J}$. Although spin is a purely quantum mechanical concept with no true analog in classical physics, an introduction of "classical spin" $\boldsymbol{S}$ helps to understand the structure of equations and some details of spin dynamics. In the next chapter the classical spins will be associated with the local magnetic moments of the magneto-ordered material.

The dynamic equation for the dimensionless spin variable has the form

$$i\frac{d}{dt}\boldsymbol{S} = \left[\boldsymbol{S}, \frac{\mathcal{H}}{\hbar}\right]_c , \tag{1.26}$$

with the following spin commutation rules:

$$\begin{aligned} [S^x, S^y]_c &= iS^z , \\ [S^y, S^z]_c &= iS^x , \\ [S^z, S^x]_c &= iS^y . \end{aligned} \tag{1.27}$$

Classically, the magnetic dipole moment $\boldsymbol{\mu}$ appears as a result of the rotation of charged particle. It equals a constant times the angular momentum

$$\boldsymbol{\mu} = -\hbar\gamma\, \boldsymbol{S} , \tag{1.28}$$

where γ is the so-called gyromagnetic ratio, and the negative sign corresponds to electronic spins. Thus the energy of spin in the magnetic field $\boldsymbol{H} = (0, 0, H)$ is equal to

$$\mathcal{H} = \frac{(\hbar\boldsymbol{S})^2}{2I_S} + \hbar\gamma H S^z , \tag{1.29}$$

where the first term describes the kinetic energy (I_S denotes the moment of inertia) and the second term describes the interaction of magnetic dipole moment with the magnetic field $-(\boldsymbol{\mu}\cdot\boldsymbol{H})$. Taking into account that S = const., the kinetic energy $\propto \boldsymbol{S}^2$ is the integral of motion and may always be omitted.

Introducing the circular spin components $S^{\pm} = S^x \pm iS^y$, from (1.27) we get

$$\begin{aligned} [S^z, S^{\pm}]_c &= \pm S^{\pm} , \\ [S^+, S^-]_c &= 2S^z . \end{aligned} \tag{1.30}$$

From the equation of motion (1.26) and (1.29) follows

$$i\frac{d}{dt}S^- = \left[S^-, \frac{\mathcal{H}}{\hbar}\right]_c = -\omega S^- \tag{1.31}$$

and

$$i\frac{d}{dt}S^+ = \left[S^+, \frac{\mathcal{H}}{\hbar}\right]_c = \omega S^+ . \tag{1.32}$$

Their solution

$$\begin{aligned} S^- &= S^-(0)\exp(-i\omega t) , \\ S^+ &= S^+(0)\exp(i\omega t) , \end{aligned} \tag{1.33}$$

describes the spin rotation with the frequency of magnetic resonance (Figure 1.3)

$$\omega = \gamma H \, . \tag{1.34}$$

So far as

$$[S^z, \mathcal{H}]_c = 0 \, ,$$

this magnetic resonance can be excited by the alternating field applied in the x–y plane only.

The solution (1.33) for the circular spin components is similar to the solution for the complex variables (1.13). This means that the spin motion can be represented in terms of harmonic oscillator variables. Holstein and Primakoff [8] were the first who understood the convenience of this representation and found it in the form:

$$\begin{aligned} S^z &= -S + a^* a \, , \\ S^+ &= a^* \sqrt{2S - a^* a} \, , \\ S^- &= \sqrt{2S - a^* a}\, a \, . \end{aligned} \tag{1.35}$$

It is easy to check that all spin commutations (1.27) and (1.30) are valid for the commutator (1.12) expressed in terms of complex variables. The Hamiltonian (1.29) can be rewritten as

$$\mathcal{H} = \text{const} + \hbar\omega a^* a \, . \tag{1.36}$$

If we change the direction of magnetic field $H \to -H$, the transformation (1.35) becomes

$$\begin{aligned} S^z &= S - a^* a \, , \\ S^+ &= \sqrt{2S - a^* a}\, a \, , \\ S^- &= a^* \sqrt{2S - a^* a} \, . \end{aligned} \tag{1.37}$$

It is also convenient to represent the equation of motion for the spin in the form of the Landau–Lifshitz equation [14] without damping:

$$\frac{d}{dt}\boldsymbol{S} = \boldsymbol{S} \times \gamma \boldsymbol{H}_{\text{eff}} \, , \tag{1.38}$$

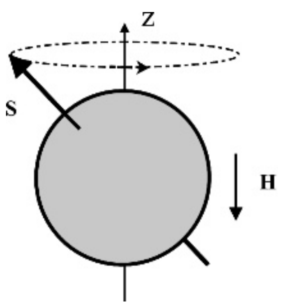

Figure 1.3 Magnetic resonance. Precession of spin in a magnetic field.

where

$$\gamma \boldsymbol{H}_{\text{eff}} = -\frac{\partial \mathcal{H}/\hbar}{\partial \boldsymbol{S}}$$

is the effective magnetic field. The absolute value of spin is conserved in (1.38) and the spin precesses around the effective field.

The principal difference between the angular momentum and spin appears when energy losses are taken into account. Friction in classical rotation leads to a decrease of both angular momentum and the rotation frequency until the system comes to a complete stop. At the same time, the absolute value of spin is conserved during the damping of spin motion with the same frequency of magnetic resonance.

1.3 Collective Variables and Harmonic Oscillators in *k*-space

Now we move to a harmonic oscillator in k-space which describes plane waves, another version of harmonic motion in nature. The equation of this oscillator has the form

$$\frac{d^2 q_k}{dt^2} + \omega_k^2 q_k = 0 \,, \tag{1.39}$$

and it differs from (1.1) by the presence of k, the wave-vector index for the generalized coordinate q_k and frequency ω_k.

1.3.1 Chain of Masses and Springs

Let us consider a mechanical system, the infinite chain containing point masses m coupled by springs (Figure 1.4a) with the same force constant κ and length d_c. This chain represents a simple model of a one-dimensional crystal, in which $x_\ell = d_\text{c} \cdot \ell$ is the equilibrium position of the ℓth mass and u_ℓ is its displacement. The equation of motion has the form:

$$m\frac{d^2 u_\ell}{dt^2} = -\kappa(2u_\ell - u_{\ell-1} - u_{\ell+1}) \,. \tag{1.40}$$

Introducing the collective variable U_k as

$$u_\ell = U_k \exp(ikx_\ell) \,, \tag{1.41}$$

where k is the wave vector, we obtain

$$\begin{aligned} m\frac{d^2 U_k}{dt^2} &= -\kappa(2 - e^{ikd_\text{c}} - e^{-ikd_\text{c}})\, U_k \\ &= -4\kappa \sin^2\left(\frac{kd_\text{c}}{2}\right) U_k \,. \end{aligned} \tag{1.42}$$

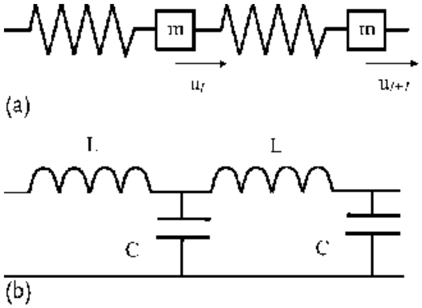

Figure 1.4 (a) Cell of the mass on a spring chain, and (b) cell of the LC chain.

Thus, the collective motion of the chain can be expressed in terms of harmonic oscillator (1.39) with $q_k = U_k$ and the frequency

$$\omega_k = 2\left|\sin\left(\frac{k d_c}{2}\right)\right|\sqrt{\frac{\kappa}{m}} \,. \tag{1.43}$$

In other words, we have obtained the spectrum ω_k of elastic waves in the chain. Note that

$$0 \le \omega_k \le \omega_b \,, \quad \omega_b \equiv 2\sqrt{\frac{\kappa}{m}} \,. \tag{1.44}$$

This indicates a band structure of excitations in the lattice system.

One can write analogous equations for the LC circuit chain (Figure 1.4b) and obtain

$$\omega_k = 2\left|\sin\left(\frac{k d_c}{2}\right)\right|\sqrt{\frac{1}{LC}} \,. \tag{1.45}$$

Problem 1.3. Write out the complex variables a_k^* and a_k for the (i) chain of masses and springs, (ii) chain of LC circuits.

1.3.2
Chain of Magnetic Particles

Consider now a chain of magnetic particles along the x-axis (Figure 1.5) with coordinates $x_\ell = d_c \cdot \ell$. Each particle represents a classical spin $\boldsymbol{S}$ oriented along the z-axis at the equilibrium. We shall write the energy of the chain in the form

$$\mathcal{H} = -\frac{\hbar\gamma H_K S}{2}\sum_j \left(\frac{S_j^z}{S}\right)^2 - \frac{J}{2}\sum_j \left(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j-1}\right) \,. \tag{1.46}$$

Here the first term describes the uniaxial anisotropy with the field H_K along the z-axis and the second term corresponds to the exchange interaction between nearest

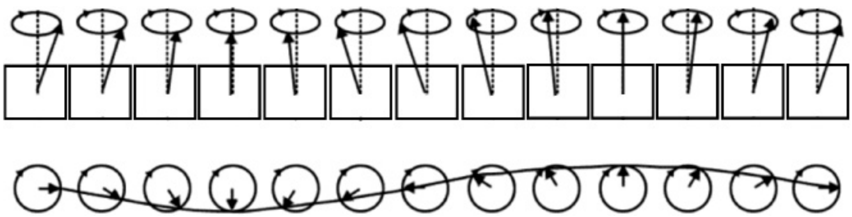

Figure 1.5 Spin wave in the chain of magnetic particles.

neighbors with the exchange integral J. In order to represent $\mathcal{H}$ in terms of harmonic oscillations, we shall use the transformation (1.37) in the approximate form

$$\begin{aligned} S_j^z &= S - a_j^* a_j \ , \\ S_j^+ &\simeq \sqrt{2S} a_j \ , \\ S_j^- &\simeq \sqrt{2S} a_j^* \ . \end{aligned} \tag{1.47}$$

Taking into account that

$$(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j-1}) = S_j^z S_{j-1}^z + \frac{1}{2}\left(S_j^+ S_{j-1}^- + S_j^- S_{j-1}^+\right) \ , \tag{1.48}$$

we can find the quadratic term of the complex variables part of (1.46):

$$\begin{aligned} \mathcal{H}^{(2)} &= \hbar\gamma H_{\mathrm{K}} \sum_j a_j^* a_j \\ &\quad + \frac{JS}{2} \sum_j \left(a_j^* a_j + a_{j-1}^* a_{j-1} - a_j^* a_{j-1} - a_{j-1}^* a_j\right) \ . \end{aligned} \tag{1.49}$$

The dynamic equation for the complex variable in this case is

$$\begin{aligned} i\frac{da_j}{dt} &= \frac{\partial \mathcal{H}^{(2)}/\hbar}{\partial a_j^*} \\ &= \gamma H_{\mathrm{K}} a_j + \frac{JS}{2}(2a_j - a_{j-1} - a_{j+1}) \ . \end{aligned} \tag{1.50}$$

Introducing the collective variable

$$a_j = a_k \exp(ikx_j) \ , \tag{1.51}$$

we simplify (1.50) to the form of a harmonic oscillator in k-space

$$i\frac{da_k}{dt} = \omega_k a_k \ , \tag{1.52}$$

where

$$\begin{aligned} \omega_k &= \gamma H_{\mathrm{K}} + \frac{JS}{2}(2 - e^{ikd_{\mathrm{c}}} - e^{-ikd_{\mathrm{c}}}) \\ &= \gamma H_{\mathrm{K}} + 2JS \sin^2\left(\frac{kd_{\mathrm{c}}}{2}\right) \end{aligned} \tag{1.53}$$

is the frequency of collective spin excitation – spin wave with wave vector k.

Once we have transformed the spin components in (1.46) to the complex variables (analog of creation and annihilation operators), we are ready to work with a system of harmonic oscillators:

$$\mathcal{H} = \sum_k \hbar\omega_k a_k^* a_k + \text{nonlinear terms.} \tag{1.54}$$

Appendix A provides additional information on the quantum mechanics of a harmonic oscillator.

Problem 1.4. Find the magnon spectrum in the chain of magnetic particles (1.46) for the case when the magnetic field $\boldsymbol{H}$ is applied along the z-axis.

Problem 1.5. Find the magnon spectrum in the chain of magnetic particles with the uniaxial anisotropy oriented along the chain.

1.4 Discussion

a) Why are the complex canonical variables so useful?
b) Is it possible to decouple two oscillators $\hbar\omega_1 a_1^* a_1$ and $\hbar\omega_2 a_2^* a_2$ with the interaction term $V(a_1^* a_2 + a_2^* a_1)$?
c) What are the similarities and differences between the angular momentum and spin?
d) Which solution is preferred (and why) for computer simulations of the magnetic dynamics in the chain of magnetic particles: (i) solution of dynamic equations for individual particles interacting with their neighbors or, (ii) solution of magnon (collective) dynamics and then application of this solution to the local magnetic moment?