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# Classical Tessellations and Three-Manifolds

With 225 Figures Including 29 Colour Illustrations



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