Contents

Part 1. A Survey of Basic Problems and Results 1

1.1 **Solvability and the Homotopy Principle** 1

1.1.1 Jets, Relations, Holonomy 1

1.1.2 The Cauchy-Riemann Relation, Oka's Principle and the Theorem of Grauert 4

1.1.3 Differentiable Immersions and the h-Principle of Smale-Hirsch . 6

1.1.4 Osculating Spaces and Free Maps 8

1.1.5 Isometric Immersions of Riemannian Manifolds and the Theorems of Nash and Kuiper 10

1.2 **Homotopy and Approximation** 13

1.2.1 Classification of Solutions by Homotopy and the Parametric h-Principle 13

1.2.2 Density of the h-Principle in the Fine Topologies 18

1.2.3 Functionally Closed Relations 22

1.3 **Singularities and Non-singular Maps** 26

1.3.1 Singularities as Differential Relations 26

1.3.2 Genericity, Transversality and Thom's Equisingularity Theorem . 30

1.4 **Localization and Extension of Solutions** 35

1.4.1 Local Solutions of Differential Relations 35

1.4.2 The h-Principle for Extensions; Flexibility and Micro-flexibility . 39

1.4.3 Ordinary Differential Equations and "Zero-Dimensional" Relations 44

1.4.4 The h-Principle for the Cauchy Extension Problem 46

Part 2. Methods to Prove the h-Principle 48

2.1 **Removal of Singularities** 48

2.1.1 Immersions and k-Mersions $V \to \mathbb{R}^q$ for $q > k$ 48

2.1.2 Immersions and Submersions $V \to W$ 52

2.1.3 Folded Maps $V^n \to W^q$ for $q \leq n$ 54

2.1.4 Singularities and the Curvature of Smooth Maps 61

2.1.5 Holomorphic Immersions of Stein Manifolds 65

2.2 **Continuous Sheaves** 74

2.2.1 Flexibility and the h-Principle for Continuous Sheaves 75
2.2.2 Flexibility and Micro-flexibility of Equivariant Sheaves 78
2.2.3 The Proof of the Main Flexibility Theorem 80
2.2.4 Equivariant Microextensions 84
2.2.5 Local Compressibility and the Proof of the Microextension Theorem 87
2.2.6 An Application: Inducing Euclidean Connections 93
2.2.7 Non-flexible Sheaves 98

2.3 **Inversion of Differential Operators** 114

2.3.1 Linearization and the Linear Inversion 114
2.3.2 Basic Properties of Infinitesimally Invertible Operators 117
2.3.3 The Nash (Newton-Moser) Process 121
2.3.4 Deep Smoothing Operators 123
2.3.5 The Existence and Convergence of Nash's Process 131
2.3.6 The Modified Nash Process and Special Inversions of the Operator $\mathscr{D}$ 139
2.3.7 Infinite Dimensional Representations of the Group Diff(V) . . . 145
2.3.8 Algebraic Solution of Differential Equations 148

2.4 **Convex Integration** 168

2.4.1 Integrals and Convex Hulls 168
2.4.2 Principal Extensions of Differential Relations 174
2.4.3 Ample Differential Relations 180
2.4.4 Fiber Connected Relations and Directed Immersions 183
2.4.5 Directed Embeddings and the Relative h-Principle 189
2.4.6 Convex Integration of Partial Differential Equations 194
2.4.7 Underdetermined Evolution Equations 195
2.4.8 Triangular Systems of P.D.E. 198
2.4.9 Isometric C^1-Immersions 201
2.4.10 Isometric Maps with Singularities 207
2.4.11 Equidimensional Isometric Maps 214
2.4.12 The Regularity Problem and Related Questions in the Convex Integration 219

Part 3. Isometric C^∞-Immersions 221

3.1 **Isometric Immersions of Riemannian Manifolds** 221

3.1.1 Nash's Twist and Approximate Immersions; Isometric Embeddings into $\mathbb{R}^q$ 221
3.1.2 Isometric Immersions $V^n \to W^q$ for $q \geqq (n+2)(n+5)/2$ 224
3.1.3 Convex Cones in the Space of Metrics 231
3.1.4 Inducing Forms of Degree $d > 2$ 232
3.1.5 Immersions with a Prescribed Curvature 235

3.1.6 Extensions of Isometric Immersions 240
3.1.7 Isometric Immersions $V^n \to W^q$ for $q \geqq (n+2)(n+3)/2$ 247
3.1.8 Isometric Cylinders $V^n \times \mathbb{R} \to W^q$ for $q \geqq (n+2)(n+3)/2$ 250
3.1.9 Non-free Isometric Maps 254

3.2 **Isometric Immersions in Low Codimension** 259

3.2.1 Parabolic Immersions 260
3.2.2 Hyperbolic Immersions 269
3.2.3 Geometric Obstructions to Isometric C^2-Immersions $V^2 \to \mathbb{R}^3$. 279
3.2.4 Isometric C^∞-Immersions $V^2 \to \mathbb{R}^q$ for $3 \leqq q \leqq 6$ 289

3.3 **Isometric C^∞-Immersions of Pseudo-Riemannian Manifolds** . . . 306

3.3.1 Local Pseudo-Riemannian Immersions 307
3.3.2 Global Immersions 312
3.3.3 Immersions with a Prescribed Curvature and the C^1-Approximation 316
3.3.4 Isotropic Maps and Non-unique Isometric Immersions 321
3.3.5 Isometric C^∞-Immersions $V^n \to W^q$ for $q \geqq [n(n+3)/2]+2$ . . . 324

3.4 **Symplectic Isometric Immersions** 327

3.4.1 Immersions of Exterior Forms 328
3.4.2 Symplectic Immersions and Embeddings 333
3.4.3 Contact Manifolds and Their Immersions 338
3.4.4 Basic Problems in the Symplectic Geometry 340

References 350

Author Index 359

Subject Index 361