

Contents

Theory of Crystal Space Groups and Infra-Red and Raman Lattice Processes of Insulating Crystals

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