

Table of Contents

Introduction	XIII
User's Manual	XVIII
Common Notations	XX
Acknowledgements	XXII

Part One. Practice and Philosophy of Constructive Mathematics 1

Chapter I. Examples of Constructive Mathematics	3
1. The Real Numbers	3
2. Constructive Reasoning	5
3. Order in the Reals	6
4. Subfields of $\mathbf{R}$ with Decidable Order	8
5. Functions from Reals to Reals	9
6. Theorem of the Maximum	10
7. Intermediate Value Theorem	11
8. Sets and Metric Spaces	12
9. Compactness	13
10. Ordinary Differential Equations	14
11. Potential Theory	15
12. The Wave Equation	17
13. Measure Theory	17
14. Calculus of Variations	19
15. Plateau's Problem	20
16. Rings, Groups, and Fields	22
17. Linear Algebra	24
18. Approximation Theory	25
19. Algebraic Topology	26
20. Standard Representations of Metric Spaces	27
21. Some Assorted Problems	30

Chapter II. Informal Foundations of Constructive Mathematics	33
1. Numbers	33
2. Operations or Rules	34
3. Sets and Presets	34

4. Constructive Proofs	35
5. Witnesses and Evidence	36
6. Logic	38
7. Functions	40
8. Axioms of Choice	42
9. Ways of Constructing Sets	43
10. Definite Presets	45

Chapter III. Some Different Philosophies of Constructive Mathematics 47

1. The Russian Constructivists	47
2. Recursive Analysis	48
3. Bishop's Constructivism	49
4. Objective Intuitionism	49
5. Sets in Intuitionism	52
6. Brouwerian Intuitionism	52
7. Martin-Löf's Philosophy	53
8. Church's Thesis	55

Chapter IV. Recursive Mathematics: Living with Church's Thesis 58

1. Constructive Recursion Theory	59
2. Diagonalization and "Weak Counterexamples"	60
3. Continuity of Effective Operations	61
4. Specker Sequences	66
5. Failure of König's Lemma: Kleene's Singular Tree	67
6. Singular Coverings	68
7. Non-Uniformly Continuous Functions	70
8. The Infimum of a Positive Function	71
9. Theorem of the Maximum Revisited	72
10. The Topology of the Disk in Recursive Mathematics	73
11. Pointwise Convergence Versus Uniform Convergence	76
12. Connectivity of Intervals	77
13. Another Surprise in Recursive Topology	77
14. A Counterexample in Descriptive Set Theory	79
15. Differential Equations with no Computable Solutions	80

Chapter V. The Role of Formal Systems in Foundational Studies 82

1. The Axiomatic Method	82
2. Informal Versus Formal Axiomatics	83
3. Adequacy and Fidelity: Criteria for Formalization	84
4. Constructive and Classical Mathematics Compared	87
5. Arithmetic of Finite Types	88
6. Formalizing Constructive Mathematics in $\mathbf{HA}^\omega$	89

Part Two. Formal Systems of the Seventies	93
Chapter VI. Theories of Rules	97
1. The Logic of Partial Terms	97
2. Combinatory Algebras	99
3. Axiomatizing Recursive Mathematics	107
4. Term Reduction and the Church-Rosser Theorem	111
5. Combinatory Logic and λ -Calculus	115
6. Term Models	119
7. Continuous Models	127
8. Finite Type Structures and Continuity in Combinatory Algebras	134
9. Set-Theoretic and Topological Models	143
10. Discussion: Adequacy and Fidelity of EON ?	145
Chapter VII. Realizability	148
1. Definition and Soundness of Realizability	148
2. Realizability and Models	152
3. Some Simple Applications	152
4. Existence Properties	156
5. q -Realizability	157
6. Rules of Choice	158
7. Discussion: Numerical Meaning	159
Chapter VIII. Constructive Set Theories	162
1. Intuitionistic Zermelo-Fraenkel Set Theory, IZF	162
2. Non-Extensional Set Theory	166
3. The Double-Negation Interpretation for IZF	173
4. Realizability for Set Theory Without Extensionality	176
5. Realizability and Models	184
6. Realizability for IZF	185
7. Connection with Realizability for Arithmetic	187
8. Consistency of Church's Thesis with IZF	188
9. The Numerical Existence Property for IZF	188
10. Discussion	190
11. The Theory B	193
12. More Discussion	196
13. Intermediate Constructive Set Theories	199
Chapter IX. The Existence Property in Constructive Set Theory	202
1. Introduction	202
2. The Set Existence Property for HAS	203
3. The Existence Property in Set Theories	206

Chapter X. Theories of Rules, Sets, and Classes	216
1. The Theory FML	216
2. The Theories EM₀ and EM₀ + J	218
3. Models of Feferman's Theories	221
4. Realizability	225
5. The Axiom of Choice	227
6. q -Realizability	229
7. Term Existence Property	232
8. Evaluation of Numerical Terms	232
9. Numerical Existence Property	233
10. Decidable Equality	233
11. Extensionality in Feferman's Theories	235
12. Some Remarks on Formalizing Mathematics in FML + DC	236
13. Functions, Operations, and Axioms of Choice	238
14. The Theory SOF (Sets, Operations, Functions)	239
15. Some Theorems of SOF	239
16. Realizability for SOF	240
17. Discussion	241
Chapter XI. Constructive Type Theories	246
1. Forms of Judgment	246
2. Philosophical Remarks on Sets, Categories, and Canonical Elements	248
3. Hypothetical Judgments	249
4. Families of Sets	250
5. Disjoint Union and Existential Quantification	250
6. Abstraction	251
7. Cartesian Product and Conjunction	251
8. The Constant E and Projection Functions	252
9. Product and Universal Quantification	253
10. Implication	254
11. N , N_k , and R	254
12. Disjoint Union	254
13. The I -Rules	255
14. How Martin-Löf's Rules Actually Define a Formal System	255
15. List of Rule Schemata of Martin-Löf's System ML₀	257
16. Other Formulations of ML₀	261
17. Interpretation of HA^ω + AC + EXT in ML₀	263
18. Formalizing Mathematics in ML₀	265
19. Is the Theory HA^ω + EXT + AC Constructive?	267
20. The Realizability Model of ML₀	268
21. Martin-Löf's Universes	275
22. The Arithmetic Theorems of Martin-Löf's Systems, and a New Realizability for Arithmetic	277
23. Discussion	279

Part Three. Metamathematical Studies	285
Chapter XII. Constructive Models of Set Theory	287
1. Aczel's "Iterative Sets"	287
2. Interpreting Subtheories of Intuitionistic ZF in Feferman's Theories	296
Chapter XIII. Proof-Theoretic Strength	300
1. Definitions About Proof-Theoretic Strength	300
2. $\Sigma_1^1 - \mathbf{AC}$ and $\mathbf{ID}_1^-$	303
3. The Strength of Feferman's Theory $\mathbf{EM}_0 + \mathbf{J}$ and Related Theories	309
4. The Strength of Constructive Set Theory $\mathbf{T}_2$	310
5. The Strength of Martin-Löf's Theories	316
6. Theories with the Strength of Arithmetic	318
7. Conservative Extension Theorems	320
Chapter XIV. Some Formalized Metamathematics and Church's Rule	324
1. The Theories $\mathbf{Tb}$ and $\mathbf{Ta}$	324
2. Formalization of Normal-Term Arguments	327
3. Formalized Models	331
4. Church's Rule	332
5. Truth Definitions and Reflection Principles	337
6. Formalized Realizability and Existence Properties	341
7. Discussion	344
Chapter XV. Forcing	346
1. Ordinary Forcing	347
2. Conservative Extension Results	355
3. Uniform Forcing	359
Chapter XVI. Continuity	368
1. Continuity Principles	369
2. Continuity and Church's Thesis	380
3. Consistency of Brouwer's Principle and Uniform Continuity	385
4. Derived Rules Related to Continuity	390
Part Four. Metaphilosophical Studies	399
Chapter XVII. Theories of Rules and Proofs	401
1. Review of the Relevant Literature	401
2. The Main Issues	402
3. A Theory of Rules and Proofs	403

4. Undecidability of the Proof-Predicate in $\mathbf{C}$	404
5. Consistency of $\mathbf{C}$	405
6. Discussion	407
7. Frege Structures	410
8. Existence of Frege Structures	411
9. Set Theory and Frege Structures	412
10. A Theory of Rules, Proofs, and Sets	415
Historical Appendix	417
1. From Gauss to Zermelo: The Origins of Non-Constructive Mathematics	418
2. From Kant to Hilbert: Logic and Philosophy	424
3. Brouwer and the Dutch Intuitionists	430
4. Early Formal Systems for Intuitionism	432
5. Kleene: The Marriage of Recursion Theory and Intuitionism	433
6. The Russian Constructivists and Recursive Analysis	434
7. Model Theory of Intuitionistic Systems	435
8. Logical Studies of Intuitionistic Systems	435
9. Bishop and his Followers	437
10. The Latest Decade	438
References	439
Index of Axioms, Abbreviations, and Theories	451
Index of Names	455
Index of Symbols	459
Index	461