Contents

Prologue 1 Chapter 1 1 The Prehistory of the Axiom of Choice 5 1.1 Introduction 5 1.2 The Origins of the Assumption 11 1.3 The Boundary between the Finite and the Infinite 22 1.4 Cantor's Legacy of Implicit Uses 30 1.5 The Well-Ordering Problem and the Continuum Hypothesis 39 1.6 The Reception of the Well-Ordering Problem 51 1.7 Implicit Uses by Future Critics 64 1.8 Italian Objections to Arbitrary Choices 76 1.9 Retrospect and Prospect 82 Chapter 2 Zermelo and His Critics (1904–1908) 85 2.1 König's "Refutation" of the Continuum Hypothesis 86 2.2 Zermelo's Proof of the Well-Ordering Theorem 88 2.3 French Constructivist Reaction 92 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 104 2.5 The German Cantorians 108 2.6 Father and Son: Julius and Dénes König 119 2.7 An English Debate 121 2.8 Peano: Logic vs. Zermelo's Axiom 134 2.9 Brouwer: A Voice in the Wilderness	List of	Symbols				xiii
The Prehistory of the Axiom of Choice 5 1.1 Introduction 5 1.2 The Origins of the Assumption 11 1.3 The Boundary between the Finite and the Infinite 22 1.4 Cantor's Legacy of Implicit Uses 30 1.5 The Well-Ordering Problem and the Continuum Hypothesis 39 1.6 The Reception of the Well-Ordering Problem 51 1.7 Implicit Uses by Future Critics 64 1.8 Italian Objections to Arbitrary Choices 76 1.9 Retrospect and Prospect 82 Chapter 2 Zermelo and His Critics (1904–1908) 85 2.1 König's "Refutation" of the Continuum Hypothesis 86 2.2 Zermelo's Proof of the Well-Ordering Theorem 88 2.3 French Constructivist Reaction 92 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 104 2.5 The German Cantorians 108 2.6 Father and Son: Julius and Dénes König 119 2.7 An English Debate 121 2.8 Peano: Logic rs. Zermelo's Axiom 134 2.9 Brouwer: A Voice in the Wilderness 135 2.10 Enthusiasm and Mistrust in America 137	Prolog	rue				1
The Prehistory of the Axiom of Choice 5 1.1 Introduction 5 1.2 The Origins of the Assumption 11 1.3 The Boundary between the Finite and the Infinite 22 1.4 Cantor's Legacy of Implicit Uses 30 1.5 The Well-Ordering Problem and the Continuum Hypothesis 39 1.6 The Reception of the Well-Ordering Problem 51 1.7 Implicit Uses by Future Critics 64 1.8 Italian Objections to Arbitrary Choices 76 1.9 Retrospect and Prospect 82 Chapter 2 Zermelo and His Critics (1904–1908) 85 2.1 König's "Refutation" of the Continuum Hypothesis 86 2.2 Zermelo's Proof of the Well-Ordering Theorem 88 2.3 French Constructivist Reaction 92 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 104 2.5 The German Cantorians 108 2.6 Father and Son: Julius and Dénes König 119 2.7 An English Debate 121 2.8 Peano: Logic rs. Zermelo's Axiom 134 2.9 Brouwer: A Voice in the Wilderness 135 2.10 Enthusiasm and Mistrust in America 137	Chapte	er 1				
1.2 The Origins of the Assumption 1.3 The Boundary between the Finite and the Infinite 1.4 Cantor's Legacy of Implicit Uses 1.5 The Well-Ordering Problem and the Continuum Hypothesis 1.6 The Reception of the Well-Ordering Problem 1.7 Implicit Uses by Future Critics 1.8 Italian Objections to Arbitrary Choices 1.9 Retrospect and Prospect 1.9 Retrospect and Prospect Chapter 2 Zermelo and His Critics (1904–1908) 2.1 König's "Refutation" of the Continuum Hypothesis 2.2 Zermelo's Proof of the Well-Ordering Theorem 2.3 French Constructivist Reaction 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 2.5 The German Cantorians 2.6 Father and Son: Julius and Dénes König 2.7 An English Debate 2.8 Peano: Logic vs. Zermelo's Axiom 2.9 Brouwer: A Voice in the Wilderness 2.10 Enthusiasm and Mistrust in America 30 30 30 30 30 30 30 30 30 30 30 30 30						5
1.2 The Origins of the Assumption 1.3 The Boundary between the Finite and the Infinite 1.4 Cantor's Legacy of Implicit Uses 1.5 The Well-Ordering Problem and the Continuum Hypothesis 1.6 The Reception of the Well-Ordering Problem 1.7 Implicit Uses by Future Critics 1.8 Italian Objections to Arbitrary Choices 1.9 Retrospect and Prospect 1.9 Retrospect and Prospect Chapter 2 Zermelo and His Critics (1904–1908) 2.1 König's "Refutation" of the Continuum Hypothesis 2.2 Zermelo's Proof of the Well-Ordering Theorem 2.3 French Constructivist Reaction 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 2.5 The German Cantorians 2.6 Father and Son: Julius and Dénes König 2.7 An English Debate 2.8 Peano: Logic vs. Zermelo's Axiom 2.9 Brouwer: A Voice in the Wilderness 2.10 Enthusiasm and Mistrust in America 30 30 30 30 30 30 30 30 30 30 30 30 30	1.1	Introduction				5
1.3 The Boundary between the Finite and the Infinite	1.2	The Origins of the Assumption				11
1.4 Cantor's Legacy of Implicit Uses	1.3	The Boundary between the Finite and the Infinite				22
1.5 The Well-Ordering Problem and the Continuum Hypothesis	1.4	Cantor's Legacy of Implicit Uses.				30
1.6 The Reception of the Well-Ordering Problem 1.7 Implicit Uses by Future Critics		The Well-Ordering Problem and the Continuum Hypothe	sis .			39
1.7 Implicit Uses by Future Critics		The Reception of the Well-Ordering Problem				51
1.8 Italian Objections to Arbitrary Choices		Implicit Uses by Future Critics				64
Chapter 2 Zermelo and His Critics (1904–1908)		Italian Objections to Arbitrary Choices.				76
Chapter 2 Zermelo and His Critics (1904–1908) 85 2.1 König's "Refutation" of the Continuum Hypothesis 86 2.2 Zermelo's Proof of the Well-Ordering Theorem 88 2.3 French Constructivist Reaction 92 2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 104 2.5 The German Cantorians 108 2.6 Father and Son: Julius and Dénes König 119 2.7 An English Debate 121 2.8 Peano: Logic vs. Zermelo's Axiom 134 2.9 Brouwer: A Voice in the Wilderness 135 2.10 Enthusiasm and Mistrust in America 137		Retrospect and Prospect				82
2.2Zermelo's Proof of the Well-Ordering Theorem882.3French Constructivist Reaction922.4A Matter of Definitions: Richard, Poincaré, and Fréchet1042.5The German Cantorians1082.6Father and Son: Julius and Dénes König1192.7An English Debate1212.8Peano: Logic vs. Zermelo's Axiom1342.9Brouwer: A Voice in the Wilderness1352.10Enthusiasm and Mistrust in America137						85
2.2Zermelo's Proof of the Well-Ordering Theorem882.3French Constructivist Reaction922.4A Matter of Definitions: Richard, Poincaré, and Fréchet1042.5The German Cantorians1082.6Father and Son: Julius and Dénes König1192.7An English Debate1212.8Peano: Logic vs. Zermelo's Axiom1342.9Brouwer: A Voice in the Wilderness1352.10Enthusiasm and Mistrust in America137	2.1	König's "Refutation" of the Continuum Hypothesis				86
2.3 French Constructivist Reaction		Zermelo's Proof of the Well-Ordering Theorem				88
2.4 A Matter of Definitions: Richard, Poincaré, and Fréchet 2.5 The German Cantorians		French Constructivist Reaction				
2.5The German Cantorians1082.6Father and Son: Julius and Dénes König1192.7An English Debate1212.8Peano: Logic vs. Zermelo's Axiom1342.9Brouwer: A Voice in the Wilderness1352.10Enthusiasm and Mistrust in America137		A Matter of Definitions: Richard, Poincaré, and Fréchet				
2.6 Father and Son: Julius and Dénes König 119 2.7 An English Debate 121 2.8 Peano: Logic vs. Zermelo's Axiom 134 2.9 Brouwer: A Voice in the Wilderness 135 2.10 Enthusiasm and Mistrust in America 137		The German Cantorians				108
2.7 An English Debate		Father and Son: Julius and Dénes König				
2.9 Brouwer: A Voice in the Wilderness		An English Debate				121
2.9 Brouwer: A Voice in the Wilderness		Peano: Logic vs. Zermelo's Axiom				134
2.10 Enthusiasm and Mistrust in America		Brouwer: A Voice in the Wilderness				135
2.11 Retrospect and Prospect	2.10	Enthusiasm and Mistrust in America				137
- The state of the	2.11	Retrospect and Prospect			-	139

xii Contents

Chapte Zerme	er 3 lo's Axiom and Axiomatization in Transition (ıar	18_	10	19\			142
3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9	Zermelo's Reply to His Critics Zermelo's Axiomatization of Set Theory The Ambivalent Response to the Axiomatization The Trichotomy of Cardinals and Other Equivalents Steinitz and Algebraic Applications A Smoldering Controversy Hausdorff's Paradox An Abortive Attempt to Prove the Axiom of Choice Retrospect and Prospect							143 150 160 167 171 175 185 188
Chapte	er 4							
	arsaw School, Widening Applications,							
	s of Set Theory (1918–1940)							196
4.1	A Survey by Sierniński							107
4.2	A Survey by Sierpiński Finite, Infinite, and Mediate	•	٠	•	٠			197
4.3	Cardinal Equivalents	•	•	•	٠	•	٠	209
4.4	Zorn's Lemma and Related Principles	•	•	٠	•	٠	٠	213 220
4.5	Widening Applications in Algebra		•	٠				220
4.6	Convergence and Compactness in General Topology	•	•	•	٠		٠	235
4.7	Negations and Alternatives	•	•	•	٠	٠	٠	242
4.8	Negations and Alternatives The Axiom's Contribution to Logic.	•	•	•	•	•	٠	242
4.9	Shifting Axiomatizations for Set Theory	•	•	•			٠	260
	Consistency and Independence of the Axiom	•	•	-	•	٠	•	272
4 1 1	Scepticism and Inquiry	•	•	•	•	•	•	284
4.12	Retrospect and Prospect		•					290
Epilog	ue: After Gödel							293
-F8		•	•	٠	•	٠	٠	293
5.1	A Period of Stability: 1940–1963		_					293
5.2	Cohen's Legacy						٠	301
Conclu	sion							308
Appena	lix 1							
Five Le	etters on Set Theory							311
Appena								321
Journa	Abbreviations Used in the Bibliography							335
Bibliog	raphy			•	•	•	•	
				•	•	•	•	340
Comer	of Numbered Propositions			•				379
wnera	i ingev							202