

# Contents

|                                                                                                     |               |
|-----------------------------------------------------------------------------------------------------|---------------|
| <b>Introduction</b> . . . . .                                                                       | <b>1</b>      |
| <b>Notations</b> . . . . .                                                                          | <b>4</b>      |
| <br><b>Chapter 1. Elliptic Equations of Order 2: Some Standard Finite Element Methods</b> . . . . . | <br><b>6</b>  |
| 1.1. A 1-Dimensional Model Problem: The Basic Notions . . . . .                                     | 6             |
| 1.2. A 2-Dimensional Problem . . . . .                                                              | 11            |
| 1.3. The Finite Element Equations . . . . .                                                         | 14            |
| 1.4. Standard Examples of Finite Element Methods . . . . .                                          | 17            |
| 1.4.1. Example 1: The $P1$ -Triangle (Courant's Triangle) . . . . .                                 | 17            |
| 1.4.2. Example 2: The $P2$ -Triangle . . . . .                                                      | 20            |
| 1.4.3. Example 3: The $Q1$ -Quadrangle . . . . .                                                    | 23            |
| 1.4.4. Example 4: The $Q2$ -Quadrangle . . . . .                                                    | 24            |
| 1.4.5. A Variational Crime: The $P1$ Nonconforming Element . . . . .                                | 25            |
| 1.5. Mixed Formulation and Mixed Finite Element Methods for Elliptic Equations . . . . .            | 27            |
| 1.5.1. The One Dimensional Problem . . . . .                                                        | 28            |
| 1.5.2. A Two Dimensional Problem . . . . .                                                          | 32            |
| <br><b>Chapter 2. Upwind Finite Element Schemes</b> . . . . .                                       | <br><b>37</b> |
| 2.1. Upwind Finite Differences . . . . .                                                            | 37            |
| 2.2. Modified Weighted Residual (MWR) . . . . .                                                     | 41            |
| 2.3. Reduced Integration of the Advection Term . . . . .                                            | 43            |
| 2.4. Computation of Directional Derivatives at the Nodes . . . . .                                  | 48            |
| 2.5. Discontinuous Finite Elements and Mixed Interpolation . . . . .                                | 50            |
| 2.6. The Method of Characteristics in Finite Elements . . . . .                                     | 55            |
| 2.7. Perturbation of the Advective Term: Bredif (1980) . . . . .                                    | 57            |

|                                                                                                                        |                |
|------------------------------------------------------------------------------------------------------------------------|----------------|
| 2.8. Some Numerical Tests and Further Comments . . . . .                                                               | 58             |
| 2.8.1. One Dimensional Stationary Advection Equation<br>(56) . . . . .                                                 | 58             |
| 2.8.2. Two Dimensional Stationary Advection Equation . . . . .                                                         | 59             |
| 2.8.3. Time Dependent Advection . . . . .                                                                              | 68             |
| <br><b>Chapter 3. Numerical Solution of Stokes Equations . . . . .</b>                                                 | <br><b>72</b>  |
| 3.1. Introduction . . . . .                                                                                            | 72             |
| 3.2. Velocity–Pressure Formulations: Discontinuous<br>Approximations of the Pressure . . . . .                         | 75             |
| 3.2.1. $u_h$ : $P1$ Nonconforming Triangle (§1-4-5); $p_h$ :<br>Piecewise Constant . . . . .                           | 76             |
| 3.2.2. $u_h$ : $P2$ Triangle $p_h$ : $P0$ (= Piecewise Constant) . . . . .                                             | 79             |
| 3.2.3. $u_h$ : “ $P2$ +bubble” Triangle (or Modified $P2$ );<br>$p_h$ : Discontinuous $P1$ . . . . .                   | 79             |
| 3.2.4. $u_h$ : $Q2$ Quadrangle; $p_h$ : $Q1$ Discontinuous . . . . .                                                   | 80             |
| 3.2.5. Numerical Solution by Penalty Methods . . . . .                                                                 | 81             |
| 3.2.6. Numerical Results and Further Comments . . . . .                                                                | 85             |
| 3.3. Velocity–Pressure Formulations: Continuous<br>Approximation of the Pressure and Velocity . . . . .                | 86             |
| 3.3.1. Introduction . . . . .                                                                                          | 86             |
| 3.3.2. Examples and Error Estimates . . . . .                                                                          | 86             |
| 3.3.3. Decomposition of the Stokes Problem . . . . .                                                                   | 88             |
| 3.4. Vorticity–Pressure–Velocity Formulations: Discontinuous<br>Approximations of Pressure and Velocity . . . . .      | 93             |
| 3.5. Vorticity Stream-Function Formulation: Decompositions<br>of the Biharmonic Problem . . . . .                      | 97             |
| <br><b>Chapter 4. Navier-Stokes Equations: Accuracy Assessments<br/>and Numerical Results . . . . .</b>                | <br><b>101</b> |
| 4.1. Remarks on the Formulation . . . . .                                                                              | 101            |
| 4.2. A review of the Different Methods . . . . .                                                                       | 103            |
| 4.2.1 Velocity–Pressure Formulations: Discontinuous<br>Approximations of the Pressure . . . . .                        | 103            |
| 4.2.2. Velocity–Pressure Formulations: Continuous<br>Approximations of the Pressure . . . . .                          | 104            |
| 4.2.3. Vorticity–Pressure–Velocity Formulations:<br>Discontinuous Approximations of Pressure and<br>Velocity . . . . . | 104            |
| 4.2.4. Vorticity Stream-Function Formulation . . . . .                                                                 | 106            |

|                                                                                                                                  |            |
|----------------------------------------------------------------------------------------------------------------------------------|------------|
| 4.3. Some Numerical Tests . . . . .                                                                                              | 107        |
| 4.3.1. The Square Wall Driven Cavity Flow . . . . .                                                                              | 107        |
| 4.3.2. An Engineering Problem: Unsteady 2-D Flow<br>Around and In an Air-Intake . . . . .                                        | 116        |
| <b>Chapter 5. Computational Problems and Bookkeeping . . . . .</b>                                                               | <b>120</b> |
| 5.1. Mesh Generation . . . . .                                                                                                   | 120        |
| 5.2. Solution of the Nonlinear Problems . . . . .                                                                                | 123        |
| 5.2.1. Successive Approximations (or Linearization) with<br>Under Relaxation . . . . .                                           | 123        |
| 5.2.2. Newton-Raphson Algorithm . . . . .                                                                                        | 124        |
| 5.2.3. Conjugate Gradient Method (with Scaling) for<br>Nonlinear Problems . . . . .                                              | 125        |
| 5.2.4. A Splitting Technique for the Transient Problem . . . . .                                                                 | 127        |
| 5.3. Iterative and Direct Solvers of Linear Equations . . . . .                                                                  | 127        |
| 5.3.1. Successive Over Relaxation . . . . .                                                                                      | 127        |
| 5.3.2. Cholesky Factorizations . . . . .                                                                                         | 128        |
| 5.3.3. Out of Core Factorizations . . . . .                                                                                      | 131        |
| 5.3.4. Preconditioned Conjugate Gradient . . . . .                                                                               | 133        |
| <b>Appendix 1. The Patch Test of the <math>P_1</math> Nonconforming<br/>    Triangle: Sketchy Proof of Convergence . . . . .</b> | <b>136</b> |
| <b>Appendix 2. Numerical Illustration . . . . .</b>                                                                              | <b>138</b> |
| <b>Appendix 3. The Zero Divergence Basis for 2-D <math>P_1</math> Nonconforming<br/>    Elements . . . . .</b>                   | <b>142</b> |
| Three Dimensional Case . . . . .                                                                                                 | 144        |
| <b>References . . . . .</b>                                                                                                      | <b>148</b> |
| <b>Index . . . . .</b>                                                                                                           | <b>161</b> |