

# Contents

<b>Lecture I</b>	
<b>The Early History of Fermat's Last Theorem</b>	<b>1</b>
1 The Problem	1
2 Early Attempts	4
3 Kummer's Monumental Theorem	6
4 Regular Primes	9
5 Kummer's Work on Irregular Prime Exponents	10
6 Other Relevant Results	12
7 The Golden Medal and the Wolfskehl Prize	13
Bibliography	16
<b>Lecture II</b>	
<b>Recent Results</b>	<b>19</b>
1 Stating the Results	19
2 Explanations	21
Bibliography	29
<b>Lecture III</b>	
<b>B.K. = Before Kummer</b>	<b>35</b>
1 The Pythagorean Equation	36
2 The Biquadratic Equation	37
3 The Cubic Equation	39
4 The Quintic Equation	45
5 Fermat's Equation of Degree Seven	46
Bibliography	47

<b>Lecture IV</b>	
<b>The Naïve Approach</b>	<b>51</b>
1 The Relations of Barlow and Abel	51
2 Sophie Germain	54
3 Congruences	57
4 Wendt's Theorem	61
5 Abel's Conjecture	63
6 Fermat's Equation with Even Exponent	65
7 Odds and Ends	69
Bibliography	70
<b>Lecture V</b>	
<b>Kummer's Monument</b>	<b>75</b>
1 A Justification of Kummer's Method	75
2 Basic Facts about the Arithmetic of Cyclotomic Fields	77
3 Kummer's Main Theorem	82
Bibliography	90
<b>Lecture VI</b>	
<b>Regular Primes</b>	<b>93</b>
1 The Class Number of Cyclotomic Fields	93
2 Bernoulli Numbers and Kummer's Regularity Criterion	99
3 Various Arithmetic Properties of Bernoulli Numbers	103
4 The Abundance of Irregular Primes	106
5 Computation of Irregular Primes	107
Bibliography	111
<b>Lecture VII</b>	
<b>Kummer Exits</b>	<b>115</b>
1 The Periods of the Cyclotomic Equation	115
2 The Jacobi Cyclotomic Function	117
3 On the Generation of the Class Group of the Cyclotomic Field	119
4 Kummer's Congruences	120
5 Kummer's Theorem for a Class of Irregular Primes	126
6 Computations of the Class Number	130
Bibliography	134
<b>Lecture VIII</b>	
<b>After Kummer, a New Light</b>	<b>139</b>
1 The Congruences of Mirimanoff	139
2 The Theorem of Krasner	148
3 The Theorems of Wieferich and Mirimanoff	151
4 Fermat's Theorem and the Mersenne Primes	154
5 Summation Criteria	155
6 Fermat Quotient Criteria	159
Bibliography	161

Lecture IX	
The Power of Class Field Theory	165
1 The Power Residue Symbol	165
2 Kummer Extensions	167
3 The Main Theorems of Furtwängler	168
4 The Method of Singular Integers	170
5 Hasse	172
6 The $p$ -Rank of the Class Group of the Cyclotomic Field	178
7 Criteria of $p$ -Divisibility of the Class Number	184
8 Properly and Improperly Irregular Cyclotomic Fields	188
Bibliography	193
Lecture X	
Fresh Efforts	199
1 Fermat's Last Theorem Is True for Every Prime Exponent Less Than 125000	199
2 Euler Numbers and Fermat's Theorem	202
3 The First Case Is True for Infinitely Many Pairwise Relatively Prime Exponents	205
4 Connections between Elliptic Curves and Fermat's Theorem	207
5 Iwasawa's Theory	209
6 The Fermat Function Field	211
7 Mordell's Conjecture	214
8 The Logicians	216
Bibliography	218
Lecture XI	
Estimates	225
1 Elementary (and Not So Elementary) Estimates	225
2 Estimates Based on the Criteria Involving Fermat Quotients	229
3 Thue, Roth, Siegel and Baker	232
4 Applications of the New Methods	237
Bibliography	241
Lecture XII	
Fermat's Congruence	245
1 Fermat's Theorem over Prime Fields	245
2 The Local Fermat's Theorem	251
3 The Problem Modulo a Prime-Power	253
Bibliography	260
Lecture XIII	
Variations and Fugue on a Theme	263
1 Variation I (In the Tone of Polynomial Functions)	263
2 Variation II (In the Tone of Entire Functions)	265

3 Variation III (In the Theta Tone)	266
4 Variation IV (In the Tone of Differential Equations)	270
5 Variation V (Giocososo)	271
6 Variation VI (In the Negative Tone)	272
7 Variation VII (In the Ordinal Tone)	273
8 Variation VIII (In a Nonassociative Tone)	273
9 Variation IX (In the Matrix Tone)	275
10 Fugue (In the Quadratic Tone)	277
Bibliography	286
Epilogue	291
Index of Names	299
Subject Index	305