

Contents

1	Introduction to Fractional Calculus.....	1
1.1	Introduction	1
1.2	Birth of Fractional Calculus.....	2
1.3	Fractional Calculus a Generalization of Integer Order Calculus	3
1.4	Historical Development of Fractional Calculus.....	5
1.4.1	The Popular Definitions of Fractional Derivatives/Integrals in Fractional Calculus	10
1.4.1.1	Riemann-Liouville	10
1.4.1.2	Grunwald-Letnikov: (Differintegrals).....	11
1.4.1.3	M. Caputo (1967).....	11
1.4.1.4	Oldham and Spanier (1974).....	11
1.4.1.5	K.S. Miller and B. Ross (1993).....	11
1.4.1.6	Kolwankar and Gangal (1994).....	11
1.5	About Fractional Integration Derivatives and Differintegration.....	12
1.5.1	Fractional Integration Riemann-Liouville (RL).....	12
1.5.2	Fractional Integration Weyl's (W).....	14
1.5.3	Nature of Kernel for Fractional Integration	14
1.5.4	Fractional Derivatives Riemann-Liouville (RL) Left Hand Definition (LHD)	15
1.5.5	Fractional Derivatives Caputo Right Hand Definition (RHD).....	17
1.5.6	Fractional Derivatives of Same Order but Different Types RL-Caputo.....	19
1.5.7	Fractional Differintegrals Grunwald Letnikov (GL).....	20
1.5.8	Fractional Derivative Weyl's.....	21
1.5.9	Scale Invariance and Power Law	22
1.5.10	Fourier Transform of Fractional Derivative.....	25
1.5.11	Composition and Property	26
1.5.12	Fractional Derivative for Some Standard Function.....	28
1.6	Solution of Fractional Differential Equations.....	30
1.6.1	Abel's Fractional Integral Equation of Tautochrone.....	31
1.6.2	Fractional Damped Motion	34
1.6.3	Formal Definition of Fractional Differential and Fractional Integral Equation.....	35
1.7	Fractional Calculus and Law of Irreversibility Non-locality	37
1.8	Stable Random Variables and Generalization of Normal Probability Density Function	38
1.9	Conservation of Probability	40

1.10	Half Order Fractional Differentiation Embedded in Standard Fick's Law and Its Extension to Describe Anomalous Diffusion	44
1.11	Fractional Brownian Motion	46
1.12	A Thought Experiment	48
1.13	Quotable Quotes about Fractional Calculus	50
1.14	Concluding Comments	50
2	Functions Used in Fractional Calculus	51
2.1	Introduction	51
2.2	Functions for the Fractional Calculus	51
2.2.1	Gamma Function	51
2.2.1.1	Representation of Gamma Function	52
2.2.1.2	Basic Properties of Gamma Function	52
2.2.2	Hypergeometric Functions	60
2.2.3	Mittag-Leffler Function	61
2.2.3.1	One-Parameter Mittag-Leffler Function	62
2.2.3.2	Two Parameter Mittag-Leffler Functions	63
2.2.3.3	Variants of Mittag-Leffler Function	65
2.2.3.4	Laplace Transforms of Mittag-Leffler Function	66
2.2.4	Agarwal Function	67
2.2.5	Erdelyi's Function	67
2.2.6	Robotnov-Hartley Function	68
2.2.7	Miller Ross Function	68
2.2.8	Generalized Cosine and Sine Function	71
2.2.9	Generalized R Function and G Function	73
2.2.9.1	Relation to Elementary Functions	74
2.2.9.2	Relationship of R Function to Other Generalized Function	74
2.2.9.3	Further Generalized Function (G Function)	75
2.2.10	Bessel Function	75
2.3	List of Laplace and Inverse Laplace Transforms Related to Fractional Calculus	77
2.4	Paradoxial Conditions for Using Generalized Differentiation and Integration Expressions and Cautions	81
2.5	Non-exponential Relaxation Power Law and Memory Integrals	83
2.6	Boltzmann's Superposition Principle	86
2.7	Motivation to Use Higher Transcendental Functions to Solve Fractional Differential Equations	87
2.8	Fractional Derivatives and Integrals of Important Functions with Use of Higher Transcendental Functions	89
2.9	Irregular Functions and Measure of Irregularity (Roughness) with Box Dimmension, Holder and Hurst's Exponents	92
2.9.1	Measure of Roughness of Graph	93
2.9.2	Generation of Irregular Graph	94
2.9.3	Determination of Box-Dimension of an Irregular Graph	95

2.9.4	Difference in Persistent Anti Persistent Noise and Motion from Power law of Power Spectral Density	97
2.10	Concluding Comments	98
3	Observation of Fractional Calculus in Physical System Description	101
3.1	Introduction	101
3.2	Temperature Heat Flux Relationship for Heat Flowing in Semi-infinite Conductor	102
3.3	Single Thermocouple Junction Temperature in Measurement of Heat Flux	104
3.4	Heat Transfer	107
3.5	Driving Point Impedance of Semi-Infinite Lossy Transmission Line	110
3.5.1	Practical Application of the Semi-Infinite Line in Circuits	116
3.5.1.1	Semi-integrator Circuit	116
3.5.1.2	Semi-differentiator Circuit	118
3.5.2	Application of Fractional Integral and Fractional Differentiator Circuit in Control System	120
3.5.3	Bode's Integrals	122
3.6	Semi Infinite Lossless Transmission Line	124
3.7	Partial Differential Equations and Operational Calculus	130
3.8	Fick's Diffusion Discussion	132
3.9	Cattaneo Diffusion	137
3.10	Anomalous Diffusion	139
3.11	Truncation of Semi-Infinite System to a Finite System	140
3.12	Approximating the Half Order by Self Similar Structure and Its Relation to Continued Fraction Expansion	143
3.13	Dynamics of Chain Network	147
3.14	Dynamics of Charged Chain Network in Electric Field	153
3.15	Concluding Comments	156
4	Concept of Fractional Divergence and Fractional Curl	157
4.1	Introduction	157
4.2	Concept of Fractional Divergence for Particle Flux	157
4.3	Fractional Kinetic Equation	159
4.4	Discrete Difference and Continuum Limit and Differential Operator in Random Walk Context	162
4.4.1	Integer Order Discrete Difference and Continuum Limit and Differential Operator	162
4.4.2	Fractional Order Discrete Difference and Continuum Limit and Fractional Differential Operator	163
4.4.3	Fourier Representation of Fractional Difference and Derivative	165
4.4.4	Stochastic Fractional Difference Equations	166

4.4.5	Random Walker with Memory Concept of Persistence and Anti-persistence Walk with Long Memory and Short Term Memory.....	170
4.5	Nuclear Reactor Neutron Flux Description	172
4.6	Classical Constitutive Neutron Diffusion Equation.....	173
4.6.1	Discussion on Classical Constitutive Equations	174
4.6.2	Graphical Explanation	175
4.6.3	About Surface Flux Curvature.....	176
4.6.4	Statistical and Geometrical Explanation for Non-local Divergence	177
4.6.5	Point Kinetic Equation in Heterogeneous Background.....	178
4.6.6	Revisiting the Realm of Brownian Motion	181
4.6.7	The Continuous Time Random Walk (CTRW) Model.....	182
4.7	Diffusion with Long Rests	184
4.8	Diffusion with Long Jumps	186
4.9	Fractional Divergence in Neutron Diffusion Equations.....	190
4.9.1	Solution of Classical Constitutive Neutron Diffusion Equation (Integer Order)	192
4.9.2	Solution of Fractional Divergence Based Neutron Diffusion Equation (Fractional Order)	193
4.9.3	Fractional Geometrical Buckling and Non-point Reactor Kinetics.....	194
4.9.4	Fractional Reactor Kinetic Equation.....	195
4.9.5	Growth of Neutron Flux with Time for Different Values of Fractional Orders and Fractional Criticality	199
4.10	Concept of Fractional Curl in Electromagnetics	200
4.10.1	Concept of Chirality.....	200
4.10.2	Duality of Solutions	200
4.10.3	Fractional Curl Operator	201
4.10.4	Wave Propagation in Unbounded Chiral Medium.....	201
4.10.5	Reflection in Chiral Medium	203
4.10.6	Transverse Wave Impedance	205
4.10.7	Propagation of Electromagnetic Waves in Bi-isotropic Medium.....	207
4.10.8	Fractional Non-symmetric Transmission Line.....	208
4.10.9	Input Impedance of Terminated Fractional Non-symmetric Line	209
4.11	Concluding Comments	210
5	Fractional Differintegrations Insight Concepts	213
5.1	Introduction	213
5.2	Calculating Fractional Integral	213
5.2.1	Existence of Fractional Differintegration.....	214
5.2.2	Useful Procedure for Calculating Fractional Integral	215
5.2.3	Calculating Fractional Integral with Non-zero Lower Limit	217
5.2.4	Fractional Integral for Analytical Function.....	217

5.3	Fractional Differintegration of Product of Two Functions	218
5.4	Symbol Standardization and Description for Differintegration	221
5.5	Riemann-Liouville Fractional Differintegral	222
5.5.1	Scale Transformation	222
5.5.2	Changing Shape of Curve While Obtaining Fractional Integration and Differentiation	225
5.5.3	Homogeneous and Heterogeneous Scales in Fractional Integration/Differentiation	226
5.5.4	Convolution Example	227
5.5.5	Practical Example of RL Differintegration in Electrical Circuit Element Description	231
5.6	Grunwald-Letnikov Fractional Differintegration	234
5.7	Unification of Differintegration through Binomial Coefficients	237
5.8	Short Memory Principle- A Moving Start Point Approximation and Its Error	240
5.9	Matrix Approach to Discretize Fractional Differintegration and Weights	242
5.10	Use of Discrete Fractional Order Differintegration in Fractional Order Signal Processing	244
5.11	Infinitesimal Element Geometrical Interpretation of Fractional Differintegrations	247
5.11.1	Integration	247
5.11.2	Differentiation	249
5.12	Local Fractional Derivatives (LFD)	250
5.12.1	KG- LFD for Order Less Than Unity	251
5.12.2	KG- LFD for Order Greater Than Unity	252
5.12.3	Critical Order of a Function and Its Relation to the Box Dimension	252
5.12.4	Information Content in LFD	255
5.12.5	Finding Holder Exponent for Singularity at a Point	259
5.13	Numerical Solution of Fractional Order Differential Equation by Use of Grunwald-Letnikov Technique	260
5.13.1	The Algorithm	260
5.13.2	Obtaining the Step Response	261
5.13.3	Fractional Order System and Integer Order System Comparison	261
5.13.3.1	Order of the FOS- n	261
5.13.3.2	Significance of Parameters a and b	262
5.13.3.3	Effect of Initial Conditions	264
5.14	Line, Surface and Volume Integration of Fractal Distributions	265
5.15	Fractional Generalization of Gauss's Law and Stroke's Law	268
5.16	Concluding Comments	269
6	Initialized Differintegrals and Generalized Calculus	271
6.1	Introduction	271
6.2	Notations of Differintegrals	272
6.3	Requirement of Initialization	273

6.4	Initialization Fractional Integration (Riemann-Liouville Approach).....	274
6.4.1	Terminal Initialization	275
6.4.2	Side-Initialization.....	276
6.5	Initializing Fractional Derivative (Riemann-Liouville Approach).....	277
6.5.1	Terminal Initialization	278
6.5.2	Side-Initialization.....	279
6.6	Initializing Fractional Differintegrals (Grunwald-Letnikov Approach)	280
6.7	Properties and Criteria for Generalized Differintegrals	282
6.7.1	Terminal Charging	285
6.7.2	Side-Charging	286
6.8	Initialization with Caputo Derivative and Its Difficulties.....	286
6.8.1	Relation between Caputo and Riemann-Liouville (RL) Fractional Derivative and Issues Relating to Initialization	287
6.8.2	Un-Initialized Derivatives RL and Caputo	288
6.8.3	Evaluation of RL and Caputo Derivative from the Start Point of the Function.....	291
6.8.4	Initialization of Caputo Derivative	292
6.8.5	Generalization of RL and Caputo Formulations	298
6.8.6	Observations Regarding Difficulties in Caputo Initialization and Demanding Physical Conditions vis-à-vis RL Initialization Conditions and Relation to Physics in Solving Fractional Order Differential Equations.....	300
6.9	Fractional Differintegrations for Periodic Signals	301
6.9.1	Fractional Derivative/Integral of Generalized Periodic Function	301
6.9.2	Fractional Derivative of Periodic Function with Lower Terminal Not at Minus Infinity	303
6.10	Fractional Advection Dispersion Equation and Its Solution.....	305
6.11	Identification of Random Delays	307
6.11.1	Random Delay a Stochastic Behavior.....	307
6.11.2	About Levy Distribution	309
6.11.3	Fractional Stochastic Dynamic Model	311
6.11.4	Fractional Delay Dynamics.....	317
6.11.5	The Random Dynamics of Computer Control System.....	320
6.12	Concluding Comments	321
7	Generalized Laplace Transform for Fractional Differintegrals.....	323
7.1	Introduction	323
7.2	Recalling Laplace Transform Fundamentals	323
7.3	Laplace Transform of Fractional Integrals.....	329
7.3.1	Decomposition of Fractional Integral in Integer Order.....	330
7.3.2	Decomposition of Fractional Order Integral in Fractional Order	334
7.4	Laplace Transformation of Fractional Derivatives	336

7.4.1	Decomposition of Fractional Order Derivative in Integer Order	338
7.4.2	Decomposition of Fractional Derivative in Fractional Order	342
7.4.3	Effect of Terminal Charging on Laplace Transforms	343
7.5	Start Point Shift Effect.....	344
7.5.1	Fractional Integral.....	344
7.5.2	Fractional Derivative	345
7.6	Laplace Transform of Initialization Function	345
7.6.1	Fractional Integral.....	345
7.6.2	Fractional Derivative	346
7.7	Examples of Initialization in Fractional Differential Equations	346
7.8	The Fundamental Fractional Order Differential Equation	350
7.8.1	The Generalized Impulse Response Function.....	351
7.9	Problem of Scalar Initialization	355
7.10	Problem of Vector Initialization	357
7.11	Laplace Transform $s \rightarrow w$ Plane for Fractional Controls Stability....	360
7.12	Rational Approximations of Fractional Laplace Operator.....	362
7.12.1	Finding Arbitrary Root of Polynomial Approximation for Fractional Laplace Operator.....	363
7.12.2	Fractional Power Pole and Fractional Power Zero to Approximate Fractional Laplace Operator.....	364
7.12.2.1	Singularity Structure for a Single Fractional Power Pole (FPP).....	365
7.12.2.2	Geometrical Derivation of Recurring Relationship of Fractional Power Pole for Fractional Integration.....	366
7.12.2.3	Recursive Algorithm for Fractional Power Pole	368
7.12.2.4	Singularity Structure for a Single Fractional Power Zero (FPZ)	370
7.13	Realization of Constant Phase Element	371
7.13.1	Asymptotic Bode Phase plot.....	372
7.13.2	Pole Zero Calculation for Constant Phase	373
7.13.3	Calculation for Pole-Zero Position of Fractional Order Impedance.....	376
7.13.4	Algorithm.....	376
7.13.5	Design and Performance of Fractional Order Impedance.....	378
7.14	Laplace Transform and Charaterization of Type of Fractional Derivative	380
7.15	Generalized Stationary Conditions	385
7.16	Concluding Comments	386
8	Application of Generalized Fractional Calculus in Electrical Circuit Analysis and Electromagnetics.....	387
8.1	Introduction	387
8.2	Electronics Operational Amplifier Circuits	387

8.2.1	Operational Amplifier Circuit with Lumped Components	387
8.2.2	Operational Amplifier Integrator with Lumped Element.....	389
8.2.3	Operational Amplifier Integrator with Distributed Element	390
8.2.4	Operational Amplifier Differential Circuit with Lumped Elements	392
8.2.5	Operational Amplifier Differentiator with Distributed Element	393
8.2.6	Operational Amplifier as Zero Order Gain with Lumped Components.....	394
8.2.7	Operational Amplifier as Zero Order Gain with Distributed Elements.....	395
8.2.8	Operational Amplifier Circuit for Semi-differintegration by Semi-infinite Lossy Line.....	396
8.2.9	Operational Amplifier Circuit for Semi-integrator	397
8.2.10	Operational Amplifier Circuit for Semi-differentiator.....	398
8.2.11	Cascaded Semi-integrators.....	399
8.2.12	Semi-integrator Series with Semi-differentiator Circuit	400
8.3	Battery Dynamics	400
8.3.1	Battery as Fractional Order System	400
8.3.2	Battery Charging Phase.....	401
8.3.3	Battery Discharge Phase	405
8.4	Tracking Filter	407
8.5	Fractional Order State Vector Representation in Circuit Theory.....	410
8.6	Realization of Fractional Order Transfer Function for $PI^\alpha D^\beta$	415
8.6.1	Fractional Order PID Controller Approximation by FPP and FPZ.....	415
8.6.2	Fractional Order Integrator	415
8.6.2.1	Rational Approximation.....	415
8.6.3	Fractional Order Differentiator	418
8.6.3.1	Rational Approximation.....	418
8.6.4	Fractional $PI^\alpha D^\beta$ Controller.....	419
8.6.4.1	Rational Approximation.....	419
8.6.5	Realization of Fractional Order Element by Circuit Network	420
8.6.5.1	Impedance Functions of a Single Port Network.....	420
8.6.5.2	Impedance Functions of a Two Port Network	421
8.6.5.3	Improved Two Port Network	421
8.7	Advance Digital Algorithms Realization for Fractional Controls	424
8.7.1	Concept of Generating Function.....	425
8.7.2	Digital Filter Realization by Rational Function Approximation for Fractional Operator	426
8.7.3	Filter Stability Consideration.....	428
8.8	Charge Conservation for Fractal Distribution.....	429
8.9	Electric Field of Fractal Distribution	430
8.9.1	Electric Field and Coulomb's Law for Fractal Distribution.....	430
8.9.2	Gauss's Law for Fractal Distribution.....	430

8.10	Magnetic Field of Fractal Distribution	431
8.10.1	Biot-Savart Law for Fractal Distribution	431
8.10.2	Ampere's Law for Fractal Distribution	432
8.11	Maxwell Equation for Fractal Distribution	432
8.12	Electric Dipole Moments for Fractal Distribution	434
8.13	Concluding Comments	436
9	Application of Generalized Fractional Calculus in Other Science and Engineering Fields	437
9.1	Introduction	437
9.2	Diffusion Model in Electrochemistry	437
9.3	Electrode-Electrolyte Interface Impedance	438
9.3.1	Normal Diffusion in a Finite Boundary System	439
9.3.2	Anomalous Diffusion in Finite Boundary System	441
9.3.2.1	Diffusion with Fractional Continuity Equation	441
9.3.2.2	Diffusion with Fractional Differential Constitutive Equation	442
9.3.2.3	Diffusion with Fractional Integral Constitutive Equation	443
9.4	Capacitor Theory	444
9.5	Fractance Circuit	446
9.6	Feedback Control System	448
9.6.1	Concept of Iso-damping	457
9.6.2	Frequency Domain Design for Fractional Order Plant and Fractional Order Controller Tuning	459
9.6.3	Family of Fractional Order Controllers	462
9.6.4	Fractional Vector Feedback Controller	462
9.6.5	Observer in Fractional Vector System	463
9.6.6	Modern Aspects of Fractional Control	465
9.7	Fractional Compensator	467
9.7.1	Generalized Compensator	467
9.7.2	Frequency Characteristics of the Lead Compensator	467
9.7.3	Compensation Using a Fractional Lead Compensator	469
9.8	Phase Shaping with Fractional Order Differ-Integrator	473
9.8.1	Application of Bode's Phase Integral	473
9.8.2	Plant with Tuned with Integer Order PID Made Iso-Damped with Additional Fractional Differ-integrator	476
9.9	Viscoelasticity (Stress-Strain)	482
9.10	Vibration Damping System	487
9.11	The Non-newtonian Fluid Anamolous Behavior with Memory	488
9.12	Concluding Comments	492
10	System Order Identification and Control	493
10.1	Introduction	493
10.2	Fractional Order Systems	493
10.3	Continuous Order Distribution	495

10.4	Determination of Order Distribution from Frequency Domain Experimental Data.....	499
10.5	Analysis of Continuous Order Distribution	501
10.6	Variable Order System.....	513
10.6.1	RL Definition for Variable Order	513
10.6.2	Laplace Transforms and Transfer Function of Variable Order System.....	515
10.6.3	GL Definition for Variable Order	517
10.7	Generalized PID-Controls.....	518
10.8	Continuum Order Feed Back Control System	520
10.9	Time Domain Response of Sinusoidal Inputs for Fractional Order Operator.....	522
10.10	Frequency Domain Response of Sinusoidal Inputs for Fractional Order Operator.....	523
10.11	Ultra-Damped System Response	524
10.12	Hyper-Damped System Response.....	525
10.13	Complex Order Differintegrations	526
10.14	Ordering the Disorder of System	531
10.14.1	Disordered Relaxation with Multiple States and Relaxation Constants.....	531
10.14.2	Appearance of Fractional Derivative in Disordered Relaxation	532
10.14.3	Generalization of Disordered Relaxation.....	533
10.14.3.1	Intermittency Disorder	534
10.14.3.2	Strong Intense Relaxation	536
10.14.3.3	Weak Intermittent Relaxation	537
10.14.3.4	Oscillating Relaxation.....	537
10.14.3.5	Generalized Dynamic Critical Index of Relaxation with intermittency	538
10.14.3.6	Spatial Disorder.....	540
10.14.3.7	Hybrid Disorder with Intermittency and Spatial Heterogeneity	541
10.15	Identification of Fractional Stochastic Processes.....	543
10.15.1	Fitting Stochastic Data into Parameters of Levy Stable Distribution	543
10.15.2	Estimation of Hurst Index by Rescaled Range (R/S Method) for Stochastic Data	545
10.16	The Concept of System Order and Disadvantage of Fractional Order System	546
10.17	Concluding Comments	548
11	Solution of Generalized Differential Equation Systems	549
11.1	Introduction.....	549
11.2	Generalized Dynamic System and Evolution of It's Solution by Principle of Action Reaction.....	550

11.3	Physical Reasoning to Solve First Order System and Its Mode Decomposition.....	551
11.4	Physical Reasoning to Solve Second Order System and Its Mode-Decomposition	555
11.5	Adomian Decomposition Fundamentals and Adomian Polynomials	558
11.6	Generalization of Physical Law of Nature Vis-À-Vis ADM	564
11.7	ADM Applied to First Order Linear Differential Equation and Mode-Decomposition Solution	565
11.8	ADM Applied to Second Order Linear Differential Equation System and Mode-Decomposition	567
11.9	ADM for First Order Linear Differential Equation System with Half Order Element and Mode-Decomposition	569
11.10	ADM for Second Order System, with Half Order Element and It's Physics	570
	11.10.1 Forcing Function as Delta Function	570
	11.10.2 Forcing Function as Step Function	572
	11.10.3 Explanation Physical Action Reaction Process Vis-À-Vis ADM.....	573
11.11	Application of Decomposition Method in RL-formulated Partial Fractional Differential Equations Linear Diffusion Wave Equation and Solution to Impulse Forcing Function	575
11.12	Generalization of Fractional Order Leading Terms in Differential Equations Formulated with Riemann-Liouville and Caputo Definitions-and Use of Integer Order Initial/Boundary Conditions-with Decomposition Method	578
	11.12.1 Decomposition of Caputo Derivative in Fractional Differential Equations	578
	11.12.2 Riemann-Liouville (RL) Derivative and Its Decomposition for Solving Fractional Differential Equation-with Integer Order Initial Condition.....	579
11.13	Application of Decomposition Method in RL Formulated Fractional Differential Equations (Non-linear) and Its Solution.....	581
11.14	Application of Decomposition Method in RL-Formulated Partial Fractional Differential Equations Non-linear Diffusion-Wave Equation and Solution	583
11.15	Decomposition Method for Generalized Equation of Motion	584
11.16	Decomposition Method for Delay Differential Equation System	586
11.17	Proposition	587
	11.17.1 Fractional Initial States-Classical Solution to FDE.....	587
	11.17.2 Basic Fractional Order Differential Equation System and its Classical Solution	590
	11.17.3 Classical Solution to Fractional Fokker-Plank Kolmogorov Equation (FFPK) by Fourier-Laplace Technique.....	591

11.17.4 Decomposition of Fractional Differential Equation
Principle-and Equivalence of RL and Caputo
Definitions to Solve FDE with Integer Order
Initial States 592

11.17.5 Application to Fractional Diffusion-Wave Equation
with Input Sine Excitation with RL-Formulation 596

11.18 Observations 597

11.19 Concluding Comments 598

References 599