

CHAPTER 0LAPLACE-BELTRAMI OPERATOR

<u>§1.</u>	<u>Riemannian manifolds</u>	12
1.1.	Covariant and contravariant vectors	12
1.2.	Metric tensor	13
1.3.	Laplace-Beltrami operator	16
<u>§2.</u>	<u>Harmonic forms</u>	18
2.1.	Differential p-forms	18
2.2.	Hodge operator	20
2.3.	Exterior derivative and coderivative	21
2.4.	Laplace-Beltrami operator	22

CHAPTER IHARMONIC FUNCTIONS

<u>§1.</u>	<u>Relations</u>	$O_w^N = O_G^N < O_{HP}^N < O_{HB}^N$	27
1.1.	Definitions		27
1.2.	Principal functions		28
1.3.	Equality of	O_w^N and O_G^N	29
1.4.	Inclusions	$O_G^N \subset O_{HP}^N \subset O_{HB}^N$	30
1.5.	Strictness		30
1.6.	Base manifold for	$N = 2$	30
1.7.	Conformal structure		31
1.8.	Reflection function		32
1.9.	Positive harmonic functions		33
1.10.	Symmetry about bisectors		34
1.11.	Relations	$O_G^N < O_{HP}^N < O_{HB}^N$	35
	NOTES TO §1		37
<u>§2.</u>	<u>Relations</u>	$O_{HB}^N < O_{HD}^N = O_{HC}^N$	37
2.1.	Inclusion	$O_{HB}^N \subset O_{HD}^N = O_{HC}^N$	37

2.2.	Strictness	38
2.3.	Case $N = 2$	38
2.4.	Poincaré N -ball B_{α}^N	40
2.5.	Representation of harmonic functions on B_{α}^N	41
2.6.	Parabolicity	43
2.7.	Asymptotic behavior of harmonic functions on B_{α}^N	44
2.8.	Characterization of O_{HP}^N and O_{HB}^N	46
2.9.	Characterization of O_{HD}^N and completion of proof	47
2.10.	Summary on harmonic functions on the Poincaré N -ball	48
2.11.	Generalization	49
2.12.	Radial harmonic functions	50
2.13.	Reduction of the problem	51
2.14.	Arbitrary harmonic functions	52
2.15.	Reduction to solution types	53
2.16.	Existence of HB functions	54
2.17.	Dirichlet integrals	55
2.18.	Existence of HD functions	56
	NOTES TO §2	57
§3.	<u>The class $O_{HL^P}^N$</u>	57
3.1.	Neither HL^P functions nor HX	58
3.2.	HX functions but no HL^P	60
3.3.	HL^P functions but no HX	64
3.4.	A test for HL^P functions	65
3.5.	HL^P functions on the Poincaré N -ball	66
	NOTES TO §3	67
§4.	<u>Completeness and harmonic degeneracy</u>	67
4.1.	Complete and degenerate or neither	68
4.2.	Not complete but degenerate	69
4.3.	Complete but nondegenerate	69
	NOTES TO §4	70

<u>§1. Quasiharmonic null classes</u>	72
1.1. Tests for quasiharmonic null classes	72
1.2. Green's functions but no QP	74
1.3. QP functions but no $QB \cup QD$	75
1.4. QD functions but no QB	76
1.5. QB functions but no QD	77
1.6. QC functions if QB and QD	78
1.7. No relations between O_{QB}^N and O_{QD}^N	78
1.8. Summary	79
NOTES TO §1	79
<u>§2. The class $O_{QL^p}^N$</u>	79
2.1. Inclusions for QL^p	79
2.2. Equalities for QL^1	80
2.3. QL^p functions but no QX	81
2.4. Neither or both QL^p and QX	82
2.5. QB functions but no QL^p	83
2.6. QD functions but no QL^p , $p > 1$	86
2.7. QL^p functions, $p > 1$, but no QL^1	88
2.8. Summary	89
NOTES TO §2	89
<u>§3. Quasiharmonic functions on the Poincaré N-ball</u>	89
3.1. Parabolicity	90
3.2. Potentials	90
3.3. Bounds for the Green's function	92
3.4. Bounds for the potential $G_{B_1}^1$	93
3.5. Bounds for the potential $G_{B_2}^1$	94
3.6. Bounds for the potential G_B^1	95
3.7. Null classes of the Poincaré 3-balls	95
3.8. Arbitrary dimension	96

3.9.	Null classes of the Poincaré N-balls	98
	NOTES TO §3	99
<u>§4.</u>	<u>Characteristic quasiharmonic function</u>	99
4.1.	Existence	100
4.2.	Characteristic property	101
4.3.	Estimating Πp_j	102
4.4.	Estimating q_1	103
4.5.	Estimating b_1	103
4.6.	Boundedness of $s(r)$	105
	NOTES TO §4	106
<u>§5.</u>	<u>Negative characteristic</u>	107
5.1.	Negative quasiharmonic functions	107
5.2.	Dependence on α	108
5.3.	Case $\alpha < -3/2$	109
5.4.	Other cases	111
5.5.	Convergence	112
5.6.	The class O_{QN}^N	113
	NOTES TO §5	114
<u>§6.</u>	<u>Integral form of the characteristic</u>	114
6.1.	Integral form	115
6.2.	Characterization of O_{QP}^N and O_{QB}^N	115
6.3.	Characterization of O_{QD}^N and O_{QC}^N	117
6.4.	Class $O_{QL^p}^N$ and the characteristic function	118
	NOTES TO §6	118
<u>§7.</u>	<u>Harmonic and quasiharmonic degeneracy of Riemannian manifolds</u>	119
7.1.	HX and QY functions, or neither	120
7.2.	HX functions but no QY	122
7.3.	HL^p functions but no QY	124
7.4.	HX functions but no QL^p	126
7.5.	HL^p functions but no QL^t	127
7.6.	QY functions but no HX	129

7.7.	QY functions but no HL^p	130
7.8.	The manifold	131
7.9.	Rate of growth of harmonic functions	133
7.10.	Exclusion of HX functions	135
7.11.	Construction of QY functions	135
	NOTES TO §7	136

CHAPTER III

BOUNDED BIHARMONIC FUNCTIONS

<u>§1.</u>	<u>Parabolicity and bounded biharmonic functions</u>	138
1.1.	Parabolic with H^2B functions	138
1.2.	Hyperbolic 2-manifolds without H^2B functions	139
1.3.	Hyperbolic space E_α^N for $N > 2$	140
1.4.	Biharmonic expansions on $E_{-3/4}^N$	143
1.5.	Exclusion of H^2B functions on $E_{-3/4}^N$	144
1.6.	Parabolic manifolds without H^2B functions	145
	NOTES TO §1	146
<u>§2.</u>	<u>Generators of bounded biharmonic functions</u>	146
2.1.	Generators on the punctured plane	147
2.2.	Biharmonic expansions on the punctured N-space	149
2.3.	Generators on the punctured 3-space	151
2.4.	Nonexistence for $N > 3$	152
	NOTES TO §2	153
<u>§3.</u>	<u>Independence on the metric</u>	153
3.1.	Radial harmonic and biharmonic functions	154
3.2.	Nonradial harmonic functions	155
3.3.	Nonradial biharmonic functions	156
3.4.	Harmonic and biharmonic expansions	158
3.5.	Nonexistence of H^2B functions for $N > 3$	159
3.6.	H^2B functions on E_α^2 and E_α^3	160
	NOTES TO §3	161

<u>§4.</u>	<u>Bounded biharmonic functions on the Poincaré N-ball</u>	162
4.1.	Characterizations	162
4.2.	Case I: $\alpha < -1$	163
4.3.	Case II: $\alpha > 3/(N - 4)$	164
4.4.	Case III: $\alpha \in (-1, 1/(N - 2))$	164
4.5.	Case IV: $\alpha \in (1/(N - 2), 3/(N - 4))$, $\alpha \neq m/(N - 2)$	164
4.6.	Case IV (continued)	166
4.7.	Case V: $\alpha \in (1/(N - 2), 3/(N - 4))$, $\alpha = m/(N - 2)$	167
4.8.	Case V (continued)	169
4.9.	Case VI: $\alpha = 1/(N - 2)$	170
4.10.	Preparation for Cases VII and VIII	171
4.11.	Case VII: $\alpha = 3/(N - 4)$	173
4.12.	Case VIII: $\alpha = -1$	174
	NOTES TO §4	177
<u>§5.</u>	<u>Completeness and bounded biharmonic functions</u>	177
5.1.	Complete but with H^2_B functions	177
5.2.	Complete and without H^2_B functions	179
5.3.	Remaining cases	179
	NOTES TO §5	179
<u>§6.</u>	<u>Bounded polyharmonic functions</u>	180
6.1.	Main Theorem	180
6.2.	Polyharmonic expansions	181
6.3.	Completion of the proof of the Main Theorem	184
6.4.	Lower dimensional spaces	185
	NOTES TO §6	186

CHAPTER IV

DIRICHLET FINITE BIHARMONIC FUNCTIONS

<u>§1.</u>	<u>Dirichlet finite biharmonic functions on the Poincaré N-ball</u>	187
1.1.	H^2_D functions on the Poincaré disk	188
1.2.	Case $\alpha = -3/4$ for $N = 2$	189
1.3.	H^2_D functions on the Poincaré N-ball	191

1.4.	Case I: $\alpha > 5/(N - 6)$	192
1.5.	Case II: $\alpha = 5/(N - 6)$	193
1.6.	Case III: $\alpha \in [1/(N - 2), 5/(N - 6))$	194
1.7.	Case IV: $\alpha < -3/(N + 2)$	196
1.8.	Case V: $\alpha = -3/(N + 2)$	196
1.9.	Test for $H^2D \neq \emptyset$	196
	NOTES TO §1	197
<u>§2.</u>	<u>Parabolicity and Dirichlet finite biharmonic functions</u>	197
2.1.	No H^2D functions on E_α^N	198
2.2.	H^2D functions on a parabolic 2-cylinder	199
2.3.	Parabolicity and H^2D degeneracy	200
2.4.	Another test for $H^2D \neq \emptyset$	201
2.5.	Original counterexample	203
2.6.	Plane with radial metrics	206
2.7.	Completion of the proof	208
	NOTES TO §2	210
<u>§3.</u>	<u>Minimum Dirichlet finite biharmonic functions</u>	210
3.1.	Existence of minimum solutions	210
3.2.	Minimum solutions as limits	211
3.3.	A nonharmonizable H^2D function	213
	NOTES TO §3	214

CHAPTER V

BOUNDED DIRICHLET FINITE BIHARMONIC FUNCTIONS

<u>§1.</u>	<u>H^2D functions but no H^2C for $N = 2$</u>	216
1.1.	Existence of H^2D functions	217
1.2.	Antisymmetric functions	219
1.3.	Main Theorem	220
1.4.	Auxiliary function μ_1	220
1.5.	Auxiliary function μ_2	221
1.6.	Auxiliary functions μ_3 through μ_6	223
1.7.	Construction of λ	224

1.8.	Characterization of $H_{\lambda}(C')$	226
1.9.	Conclusion	227
	NOTES TO §1	228
<u>§2.</u>	<u>Higher dimensions</u>	228
2.1.	Cases $N > 4$ by the Poincaré N -ball	228
2.2.	Arbitrary dimension	229
2.3.	Special cases of $f(x)G(y)$	230
2.4.	General case of $f(x)G(y)$	231
2.5.	Biharmonic functions of x	232
2.6.	Biharmonic functions $v(x)G(y)$	232
2.7.	Conclusion	234
2.8.	No relation between H^2B and H^2D degeneracies	234
	NOTES TO §2	235

CHAPTER VI

HARMONIC, QUASIHARMONIC, AND BIHARMONIC DEGENERACIES

<u>§1.</u>	<u>Harmonic and biharmonic degeneracies</u>	237
1.1.	No relations	237
1.2.	No H^2D functions	238
1.3.	No H^2B functions	240
	NOTES TO §1	240
<u>§2.</u>	<u>Corresponding quasiharmonic and biharmonic degeneracies</u>	240
2.1.	Strict inclusions	241
2.2.	H^2C functions but no QP	241
2.3.	H^2L^p functions but no QL^p	242
2.4.	Summary	244
	NOTES TO §2	244

CHAPTER VII

RIESZ REPRESENTATION OF BIHARMONIC FUNCTIONS

<u>§1.</u>	<u>Metric growth of Laplacian</u>	245
1.1.	The class H^2DC_{Δ}	246
1.2.	The class H^2CD_{Δ}	247

1.3.	Auxiliary estimates	248
1.4.	Completion of proof	249
1.5.	Application to Riesz representation	251
1.6.	Dependence on the type of R	251
	NOTES TO §1	253
<u>§2.</u>	<u>Riesz representation</u>	253
2.1.	Main result	254
2.2.	Frostman-type representation	254
2.3.	Local decomposition	257
2.4.	Energy integrals	258
2.5.	Reduction of Theorem 2.1	261
2.6.	Royden compactification	263
2.7.	Completion of the proof of Theorem 2.1	265
2.8.	H^2DD_Δ function not in H^2G	267
2.9.	Dirichlet potentials	269
	NOTES TO §2	270
<u>§3.</u>	<u>Minimum solutions as potentials</u>	271
3.1.	Preliminary considerations	271
3.2.	Rate of growth	272
3.3.	Role of QP functions	274
3.4.	Role of QB functions	275
3.5.	Nonnecessity of $R \in \tilde{O}_{QP}^N$	278
3.6.	Construction of the metric	279
	NOTES TO §3	285
<u>§4.</u>	<u>Biharmonic and (p,q)-biharmonic projection and decomposition</u>	285
4.1.	Definitions	286
4.2.	Potential p -subalgebra	287
4.3.	Energy integral	288
4.4.	The (p,q) -biharmonic projection	290
4.5.	(p,q) -quasi-harmonic classification of Riemannian manifolds	292
4.6.	Decomposition	294

4.7.	Nondegenerate manifolds	294
4.8.	Special density functions	295
4.9.	Inclusion relations	296
	NOTES TO §4	296

CHAPTER VIII

BIHARMONIC GREEN'S FUNCTION γ

<u>§1.</u>	<u>Existence criterion for γ</u>	299
1.1.	Definition	300
1.2.	Existence on N-space	301
1.3.	Biharmonic Dirichlet problem	302
1.4.	Independence	305
1.5.	Existence criterion	307
1.6.	Illustration	307
	NOTES TO §1	308
<u>§2.</u>	<u>Biharmonic measure</u>	308
2.1.	Definition	310
2.2.	Biharmonic measure on N-space	311
2.3.	Radial metric	313
2.4.	Poincaré N-ball	316
2.5.	Independence	322
2.6.	Conclusion	325
	NOTES TO §2	325
<u>§3.</u>	<u>Biharmonic Green's function γ and harmonic degeneracy</u>	326
3.1.	Alternate proof of the test for O_{γ}^N	327
3.2.	Harmonic and biharmonic Green's functions	328
3.3.	Relation to harmonic degeneracy	329
3.4.	Neither γ nor HL^p functions	332
3.5.	HL^p functions but no γ	333
3.6.	γ but no HL^p functions	333
	NOTES TO §3	335

<u>§4. Biharmonic Green's function γ and quasiharmonic degeneracy</u>	335
4.1. Existence test for QP functions	335
4.2. Strict inclusion	337
4.3. Relation to QL^p degeneracy	338
NOTES TO §4	339

CHAPTER IX

BIHARMONIC GREEN'S FUNCTION β : DEFINITION AND EXISTENCE

<u>§1. Introduction: definition and main result</u>	341
1.1. Conventional definition	341
1.2. New definition	343
1.3. Main Theorem	344
1.4. Plan of this chapter	345
NOTES TO §1	346
<u>§2. Local boundedness</u>	346
2.1. An auxiliary result	346
2.2. Locally bounded Banach space	348
2.3. Locally bounded Hilbert space	349
NOTES TO §2	350
<u>§3. Fundamental kernel</u>	350
3.1. Harmonic Green's functions	350
3.2. Fundamental kernel	351
3.3. Corresponding functional	351
3.4. Continuity	352
3.5. Auxiliary function	354
NOTES TO §3	356
<u>§4. Existence of β</u>	356
4.1. Fundamental kernel and β	356
4.2. Existence and uniqueness	356
4.3. Joint continuity	357
4.4. Existence on regular subregions	357

NOTES TO §4	359
<u>§5. β as a directed limit</u>	359
5.1. Consistency	359
5.2. Continuity	360
5.3. Convergence to zero	361
5.4. Existence only	362
NOTES TO §5	363
<u>§6. Existence of β on hyperbolic manifolds</u>	363
6.1. Hyperbolicity	363
6.2. Existence of fundamental kernel	364
6.3. Existence of β	365
NOTES TO §6	366
<u>§7. Existence of β on parabolic manifolds</u>	366
7.1. Imitation problem	366
7.2. Principal functions	367
7.3. Maximum principle	368
7.4. Generalization of Evans kernel	369
7.5. Continuity	372
7.6. Fundamental kernel	373
7.7. Unboundedness of h	375
7.8. Existence of β	375
NOTES TO §7	375
<u>§8. Examples</u>	376
8.1. Euclidean N-space	376
8.2. Dimensions 3 and 4	376
8.3. Complement of unit ball	378
8.4. Properties of K	379
8.5. Functional k_{ξ}	380
8.6. Dimension 2	381
NOTES TO §8	382

CHAPTER X

RELATION OF O_{β}^N TO OTHER NULL CLASSES

<u>§1.</u>	<u>Inclusion</u>	$O_{\beta}^N < O_{\gamma}^N$	384
1.1.	Definitions of	O_{β}^N and $\tilde{O}_{\beta}^N$	384
1.2.	Operators	β_{Ω} and γ_{Ω}	386
1.3.	Monotonicity		387
1.4.	Comparison		389
1.5.	Exhaustion		390
1.6.	Convergence of	β_{Ω}	391
1.7.	Inclusion	$O_{\beta}^N \subset O_{\gamma}^N$	392
1.8.	Elementary proof		393
1.9.	A criterion for the existence of	β	395
1.10.	Strictness of the inclusion		398
	NOTES TO §1		399
<u>§2.</u>	<u>A nonexistence test for</u>	β	399
2.1.	The class	O_{β}^N	399
2.2.	The class	$O_{SH_2}^N$	400
2.3.	The β -density	$H_{\Omega}(\cdot, y)$	401
2.4.	The β -span	S_{β}	402
2.5.	The β -density	$H(\cdot, y)$	404
2.6.	An extremum property of	$H(\cdot, y)$	406
2.7.	Proof of Theorem 2.2		408
2.8.	Plane with HD functions but no	β	408
	NOTES TO §2		409
<u>§3.</u>	<u>Manifolds with strong harmonic boundaries but without</u>	β	410
3.1.	Double of a Riemannian manifold		410
3.2.	Manifolds with HD functions but without	β	411
3.3.	Existence of HD functions		411
3.4.	Nonexistence of	β	412
	NOTES TO §3		415

<u>§4.</u>	<u>Parabolic Riemannian planes carrying β</u>	415
4.1.	Density	415
4.2.	Potentials	416
4.3.	Extremum property	417
4.4.	Consequences	418
4.5.	Green's function of the simply supported plate	419
4.6.	Kernels	420
4.7.	Strong limits	420
4.8.	Convergence of β_Ω	422
4.9.	Existence of β on C_λ	423
4.10.	Necessity	423
4.11.	Sufficiency	424
4.12.	Auxiliary formulas	425
4.13.	Case $\gamma \neq 0$	426
4.14.	Case $\gamma = 0$	430
	NOTES TO §4	431

<u>§5.</u>	<u>Further existence relations between harmonic and biharmonic</u>	
	<u>Green's functions</u>	431
5.1.	Parabolic manifolds without β	431
5.2.	Hyperbolic manifolds without β	434
5.3.	Hyperbolic manifolds with β but without γ	435
5.4.	A test for $O_\gamma^N \cap \tilde{O}_\beta^N \cap \tilde{O}_G^N$	436
5.5.	Comparison principle	437
5.6.	Expansions in spherical harmonics	437
5.7.	Main result	438
5.8.	Hyperbolicity	439
5.9.	An inequality	440
5.10.	Fourier expansion	441
5.11.	Conclusion	442
	NOTES TO §5	443

CHAPTER XI

HADAMARD'S CONJECTURE ON THE GREEN'S FUNCTION

OF A CLAMPED PLATE

<u>§1. Green's functions of the clamped punctured disk</u>	445
1.1. Clamping and simple supporting	445
1.2. Simply supported punctured disk	446
1.3. Clamped punctured disk	447
1.4. Clamped disk	447
1.5. Boundary behavior	449
1.6. Hadamard's conjecture	450
NOTES TO §1	451
<u>§2. Hadamard's problem for higher dimensions</u>	451
2.1. The manifold	452
2.2. Sign of β_0	453
2.3. Biharmonic Poisson equation	453
2.4. First inequality	454
2.5. Second inequality	455
NOTES TO §2	456
<u>§3. Duffin's function and Hadamard's conjecture</u>	456
3.1. Beta densities	457
3.2. Fundamental kernel	459
3.3. Sharpened consistency relation	460
3.4. Infinite strip	461
3.5. Negligible boundary	462
3.6. Fundamental Lemma	463
3.7. Fourier transforms	463
3.8. Completion of proof	465
3.9. Duffin's function	466
3.10. Nonconstant sign of Duffin's function	466
3.11. Additional properties	467
3.12. Biharmonic Green's potential	468

3.13. Identity of β and w	469
3.14. Counterexamples, old and new, to Hadamard's conjecture	470
NOTES TO §3	472
BIBLIOGRAPHY	473
AUTHOR INDEX	485
SUBJECT AND NOTATION INDEX	488