

Contents

Preface	v
Contents	vii
Introduction	xii
6 Basic Concepts of the Theory of Schemes	1
6.1 Affine Schemes	1
6.1.1 Localization	1
6.1.2 The Spectrum of a Ring	2
6.1.3 The Zariski Topology on $\text{Spec}(A)$	6
6.1.4 The Structure Sheaf on $\text{Spec}(A)$	8
6.1.5 Quasicoherent Sheaves	11
6.1.6 Schemes as Locally Ringed Spaces	12
Closed Subschemes	14
Sections	15
A remark	15
6.2 Schemes	16
6.2.1 The Definition of a Scheme	16
The gluing	16
Closed subschemes again	17
Annihilators, supports and intersections	18
6.2.2 Functorial properties	18
Affine morphisms	19
Sections again	19
6.2.3 Construction of Quasi-coherent Sheaves	19
Vector bundles	20
Vector Bundles Attached to Locally Free Modules	20
6.2.4 Vector bundles and GL_n -torsors.	21
6.2.5 Schemes over a base scheme S	22
Some notions of finiteness	22
Fibered products	23
Base change	28
6.2.6 Points, T-valued Points and Geometric Points	28
Closed Points and Geometric Points on varieties	32
6.2.7 Flat Morphisms	34
The Concept of Flatness	35
Representability of functors	38
6.2.8 Theory of descend	40
Effectiveness for affine descend data	43
6.2.9 Galois descend	44
A geometric interpretation	47
Descend for general schemes of finite type	48

6.2.10	Forms of schemes	48
6.2.11	An outlook to more general concepts	51
7	Some Commutative Algebra	55
7.1	Finite A-Algebras	55
7.1.1	Rings With Finiteness Conditions	58
7.1.2	Dimension theory for finitely generated k -algebras	59
7.2	Minimal prime ideals and decomposition into irreducibles	61
	Associated prime ideals	63
	The restriction to the components	63
	Decomposition into irreducibles for noetherian schemes	64
	Local dimension	65
7.2.1	Affine schemes over k and change of scalars	65
	What is $\dim(Z_1 \cap Z_2)$?	70
7.2.2	Local Irreducibility	71
	The connected component of the identity of an affine group scheme	
	G/k	72
7.3	Low Dimensional Rings	73
	Finite k -Algebras	73
	One Dimensional Rings and Basic Results from Algebraic Number	
	Theory	74
7.4	Flat morphisms	80
7.4.1	Finiteness Properties of Tor	80
7.4.2	Construction of flat families	82
7.4.3	Dominant morphisms	84
	Birational morphisms	88
	The Artin-Rees Theorem	89
7.4.4	Formal Schemes and Infinitesimal Schemes	90
7.5	Smooth Points	91
	The Jacobi Criterion	95
7.5.1	Generic Smoothness	97
	The singular locus	97
7.5.2	Relative Differentials	99
7.5.3	Examples	102
7.5.4	Normal schemes and smoothness in codimension one	109
	Regular local rings	110
7.5.5	Vector fields, derivations and infinitesimal automorphisms	111
	Automorphisms	114
7.5.6	Group schemes	114
7.5.7	The groups schemes $\mathbb{G}_a, \mathbb{G}_m$ and μ_n	116
7.5.8	Actions of group schemes	117
8	Projective Schemes	121
8.1	Geometric Constructions	121
8.1.1	The Projective Space $\mathbb{P}_A^n$	121
	Homogenous coordinates	123
8.1.2	Closed subschemes	125
8.1.3	Projective Morphisms and Projective Schemes	126
	Locally Free Sheaves on $\mathbb{P}^n$	129

	$\mathcal{O}_{\mathbb{P}^n}(d)$ as Sheaf of Meromorphic Functions	131
	The Relative Differentials and the Tangent Bundle of $\mathbb{P}_{\mathbb{S}}^n$	132
8.1.4	Separated and Proper Morphisms	134
8.1.5	The Valuative Criteria	136
	The Valuative Criterion for the Projective Space	136
8.1.6	The Construction $\text{Proj}(R)$	137
	A special case of a finiteness result.	139
8.1.7	Ample and Very Ample Sheaves	140
8.2	Cohomology of Quasicoherent Sheaves	146
8.2.1	Čech cohomology	148
8.2.2	The Künneth-formulae	150
8.2.3	The cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(r)$	151
8.3	Cohomology of Coherent Sheaves	153
	The Hilbert polynomial	157
8.3.1	The coherence theorem for proper morphisms	158
	Digression: Blowing up and contracting	159
8.4	Base Change	164
8.4.1	Flat families and intersection numbers	171
	The Theorem of Bertini	179
8.4.2	The hyperplane section and intersection numbers of line bundles	180
9	Curves and the Theorem of Riemann-Roch	183
9.1	Some basic notions	183
9.2	The local rings at closed points	185
9.2.1	The structure of $\widehat{\mathcal{O}}_{C,\mathfrak{p}}$	186
9.2.2	Base change	186
9.3	Curves and their function fields	188
9.3.1	Ramification and the different ideal	190
9.4	Line bundles and Divisors	193
9.4.1	Divisors on curves	195
9.4.2	Properties of the degree	197
	Line bundles on non smooth curves have a degree	197
	Base change for divisors and line bundles	198
9.4.3	Vector bundles over a curve	198
	Vector bundles on $\mathbb{P}^1$	199
9.5	The Theorem of Riemann-Roch	201
9.5.1	Differentials and Residues	203
9.5.2	The special case $C = \mathbb{P}^1/k$	207
9.5.3	Back to the general case	211
9.5.4	Riemann-Roch for vector bundles and for coherent sheaves.	218
	The structure of $K'(C)$	220
9.6	Applications of the Riemann-Roch Theorem	221
9.6.1	Curves of low genus	221
9.6.2	The moduli space	223
9.6.3	Curves of higher genus	234
	The "moduli space" of curves of genus g	238
9.7	The Grothendieck-Riemann-Roch Theorem	239
9.7.1	A special case of the Grothendieck -Riemann-Roch theorem	240

9.7.2	Some geometric considerations	241
9.7.3	The Chow ring	244
	Base extension of the Chow ring	247
9.7.4	The formulation of the Grothendieck-Riemann-Roch Theorem . . .	249
9.7.5	Some special cases of the Grothendieck-Riemann-Roch-Theorem .	252
9.7.6	Back to the case $p_2 : X = C \times C \longrightarrow C$	253
9.7.7	Curves over finite fields.	257
	Elementary properties of the ζ -function.	258
	The Riemann hypothesis.	261
10	The Picard functor for curves and their Jacobians	265
	Introduction:	265
10.1	The construction of the Jacobian	265
10.1.1	Generalities and heuristics :	265
	Rigidification of PIC	267
10.1.2	General properties of the functor PIC	269
	The locus of triviality	269
10.1.3	Infinitesimal properties	272
	Differentiating a line bundle along a vector field	274
	The theorem of the cube.	274
10.1.4	The basic principles of the construction of the Picard scheme of a curve.	278
10.1.5	Symmetric powers	279
10.1.6	The actual construction of the Picard scheme of a curve.	284
	The gluing	291
10.1.7	The local representability of $PIC_{C/k}^g$	294
10.2	The Picard functor on X and on J	297
	Some heuristic remarks	297
10.2.1	Construction of line bundles on X and on J	297
	The homomorphisms ϕ_M	298
10.2.2	The projectivity of X and J	301
	The morphisms ϕ_M are homomorphisms of functors	302
10.2.3	Maps from the curve C to X , local representability of $PIC_{X/k}, PIC_{J/k}$ and the self duality of the Jacobian	303
10.2.4	The self duality of the Jacobian	310
10.2.5	General abelian varieties	311
10.3	The ring of endomorphisms $\text{End}(J)$ and the ℓ -adic modules $T_\ell(J)$. .	314
	Some heuristics and outlooks	314
	The study of $\text{End}(J)$	315
	The degree and the trace	318
	The Weil Pairing	326
	The Neron-Severi groups $NS(J), NS(J \times J)$ and $\text{End}(J)$	328
	The ring of correspondences	331
10.4	Étale Cohomology	334
	The cyclotomic character.	334
10.4.1	Étale cohomology groups	335
	Galois cohomology	336
	The geometric étale cohomology groups.	338

10.4.2 Schemes over finite fields	344
The global case	346
The degenerating family of elliptic curves	350
Bibliography	357
Index	362