

Contents

1	Introduction	1
2	Hyperbolic Numbers	3
2.1	The Geometry Associated with Complex Numbers	4
2.2	Generalization of Complex Numbers	8
2.2.1	Definition of the Modulus	12
2.3	Lorentz Transformations and Space–Time Geometry	12
2.4	The Geometry Associated with Hyperbolic Numbers	15
2.4.1	Hyperbolic Rotations as Lorentz Transformations	18
2.5	Conclusions	19
2.6	Appendix	20
2.6.1	Cubic Equation and Introduction of Complex Numbers	20
2.6.2	Geometrical and Classical Definition of Hyperbolic Angles	21
	References	23
3	Geometrical Representation of Hyperbolic Numbers	25
3.1	Introduction	25
3.2	Extension of Hyperbolic Exponential Function and Hyperbolic Polar Transformation.	26
3.3	Geometry in the Hyperbolic Plane.	27
	References	31
4	Trigonometry in the Hyperbolic (Minkowski) Plane	33
4.1	Analytical Formalization of Euclidean Trigonometry	33
4.1.1	Complex Numbers Invariants in Euclidean Plane	33
4.2	Hyperbolic Rotation Invariants in Minkowski Plane	35

4.3	Extension of Hyperbolic Trigonometric Functions	36
4.3.1	Fjelstad's Extension of Hyperbolic Trigonometric Functions	38
4.4	Goniometry in the Minkowski Plane	41
4.5	Trigonometry in the Minkowski Plane	42
4.5.1	Analytical Definitions of Hyperbolic Trigonometric Functions	42
4.5.2	Trigonometric Laws in Hyperbolic Plane	44
4.6	Triangles in Hyperbolic Plane	50
4.6.1	The Triangular Property	50
4.6.2	The Elements in a Right-Angled Triangle	51
4.6.3	Solution of Hyperbolic Triangles	52
	References	55
5	Equilateral Hyperbolas and Triangles in the Hyperbolic Plane . . .	57
5.1	Theorems on Equilateral Hyperbolas	57
5.2	Triangle and Equilateral Hyperbolas	65
5.2.1	Circumscribed Hyperbola	65
5.2.2	Inscribed and Ex-inscribed Hyperbolas	67
5.2.3	Numerical Examples	71
	References	72
6	The Motions in Minkowski Space-Time (<i>Twin Paradox</i>)	73
6.1	Inertial Motions	74
6.2	Inertial and Uniformly Accelerated Motions	77
6.2.1	Uniform and Uniformly Accelerated Motions	80
6.2.2	Uniform Motions Joined by Uniformly Accelerated Motions	82
6.2.3	Reversed Triangle Inequality for Curved Lines	83
6.2.4	The Most General Uniform and Accelerated Motions	85
6.3	Non-uniformly Accelerated Motions	87
6.3.1	The Curves in Euclidean Plane	87
6.3.2	Formalization of Non-uniformly Accelerated Motions	90
6.3.3	Proper Time in Non-uniformly Accelerated Motions	93
6.4	Conclusions	94
	References	95
7	Some Final Considerations	97
	References	99
	Appendix A	101
	Index	113