CONTENTS AND ABSTRACTS

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Every topological Boolean algebra is isomorphic to a topological subalgebra of the ordered space on a non-standard model of the rationals. So in particular a countable topological boolean algebra is isomorphic to a topological subalgebra of the ordered space on an $\eta_1$ set. An atomic countable $T_3$ dense-in-itself algebra is isomorphic to a topological subalgebra of the space of rationals. (January 10, 1973)



Nonstandard techniques are used to investigate the maximal ideal space $\mathfrak{M}(D)$ of the Banach algebra $H^{\infty}(D)$ of bounded analytic functions on a planar domain D, and, more especially, to study the analytic structure in $\mathfrak{M}(D) - D$. A relatively complete discussion is presented for the unit disk, and a few results for infinitely connected domains are discussed. (August 1, 1973)



Several classical boundary value theorems including the Gross principal value theorem are proved using nonstandard methods. (August 20, 1973)



An inverse function theorem is proved under a local assumption which is weaker than differentiability in a neighborhood. (August 20, 1973)

in order to model perfectly competitive markets, i.e., markets where each trader's economic influence is negligible, we assume that the economy has $\omega$ traders, where $\omega$ is an infinite integer, and the average endowment of each trader is infinitesimal.

In these nonstandard exchange economies, we examine the relationship between outcomes obtained through bargaining, called core allocations, and the allocations arising out of the competitive price system. We show that Edgeworth's conjecture, that every core allocation is a competitive allocation, is true in nonstandard exchange economies. As a consequence of this theorem we also show that core allocations in large finite standard economies are approximately competitive allocations.

References: Brown, D. J. and A. Robinson, "Nonstandard Exchange Economies", Econometrica (to appear).

————, "A Limit Theorem on the Cores of Large Standard Exchange Economies," Proc.Nat.Acad.Sc., U.S.A., Vol.69, No. 5, 1258-1260.

Harry Gonshor

Enlargements contain various kinds of completions 60

In this paper we show that various types of completions in different senses may be obtained as subquotients of enlargements. Among the examples considered are the following: The rationals as a "completion" of the integers, the Stone-Cech compactification of a completely regular space, the second conjugate space of a Banach space, rings of quotients of rings of continuous functions, the projective cover of a compact Hausdorff space, and the completions of a Boolean algebra. The last example is studied in detail. (March 22, 1973)

C. Ward Henson and L. C. Moore, Jr.

Semi-reflexivity of the nonstandard hulls of a locally convex space 71

Let $E,F$ be vector spaces over $R$ or $C$ paired by a nonsingular bilinear form $\langle ...,... \rangle$ and let $\theta$ be a locally convex topology on $E$ which is admissible for the pairing. Given an enlargement $*\mathfrak{M}$ of a set theoretical structure $\mathfrak{M}$ which contains $E$ and $F$, let $(\hat{E},\hat{\theta})$ be the associated nonstandard hull of $(E,\theta)$. Also let $\hat{F}$ be the space of linear functionals on $\hat{E}$ which are represented by those points $q$ in $*F$ for which $\langle p,q \rangle$ is finite for all $\theta$-finite $p$ in $*E$. Then $\hat{F}$ is contained in the dual space $\hat{E}'$ of $(\hat{E},\hat{\theta})$ and $\hat{\theta}$ is an admissible topology relative to the pairing between $\hat{E}$ and $\hat{F}$.

The principal result is that $(\hat{E},\hat{\theta})$ is semi-reflexive if and only if $\hat{F}$ is $\beta(\hat{E}',\hat{E})$-dense in $\hat{E}'$. (This extends a result for normed spaces proved earlier by the authors.) Moreover, a geometric condition on $(E,\theta)$ is given which is equivalent to the semi-reflexivity of $(\hat{E},\hat{\theta})$ (and therefore this property of $(\hat{E},\hat{\theta})$ does not depend on the particular enlargement $*\mathfrak{M}$). The main technical tool is the following result which seems of interest itself.

Retraction Theorem: If $*\mathfrak{M}$ is $\kappa$-saturated and $S$ is a subspace of $\hat{E}$ which has Hamel dimension less than $\kappa$, then for each $\phi \in \hat{E}'$ there exists $\psi \in \hat{F}$ such that $\phi(x) = \psi(x)$ for all $x \in S$. (March 8, 1973)

C. Ward Henson and L. C. Moore, Jr.

Invariance of the nonstandard hulls of a uniform space 85

Let $(X,\mathfrak{U})$ be a uniform space and let $*\mathfrak{M}$ be an enlargement of some set-theoretical structure $\mathfrak{M}$ which contain $X$. It is shown that if $p \in *X$ is not $\mathfrak{U}$-pre-nearstandard and if $*\mathfrak{M}$ is $\kappa$-saturated, then the filter monad $\mu(\mathrm{Fil}(p)) = \cap\{*Y \mid p \in *Y\}$ intersects at least $\kappa$ distinct $\mathfrak{U}$-monads $\mu_{\mathfrak{U}}(q)$. As a consequence, either (i) the nonstandard hulls of $(X,\mathfrak{U})$ (constructed using the $\mathfrak{U}$-finite elements of $*X$) are all equal to the completion of $(X,\mathfrak{U})$ and are therefore independent of $*\mathfrak{M}$ or (ii) the nonstandard hulls of $(X,\mathfrak{U})$ can be made arbitrarily large by varying the choice of $*\mathfrak{M}$. Similar results hold for the classes of nonstandard hulls defined by Luxemburg. In case (i) holds, we say $(X,\mathfrak{U})$ has invariant nonstandard hulls (extending the terminology introduced by the authors for locally convex topological vector spaces).

A metric space $(X,d)$ has invariant nonstandard hulls if and only if every subset of $X$ on which every uniformly continuous function is bounded must be totally bounded. This equivalence is not true, however, for arbitrary uniform spaces. It is also shown that if $(X,\mathfrak{T})$ is a separable, topologically complete (metrizable) space, then $\mathfrak{T}$ can be defined by a metric $d$ such that $(X,d)$ is complete and has invariant nonstandard hulls. (March 8, 1973)

Joram Hirschfeld

Models of arithmetic and the semi-ring of recursive functions 99

S. Feferman, D. Scott, and S. Tennenbaum proved that every non-trivial homomorphic image of the semi-ring $R$ of recursive functions fails to be a model of arithemtic. The aim of this paper is to show that every countable model of full arithmetic can be embedded in such a homormorphic image. To prove this we modify a theorem by H. Friedman to obtain a sufficient condition for a model of arithmetic to be embeddable in a model of a fragment of arithmetic. We then introduce recursive ultrapowers - homomorphic images of $R$ which are models of that fragment. Finally, given a model, we show how to construct a recursive ultrapower which satisfies the condition of Friedman's theorem. (April 3, 1973.)

Albert E. Hurd

Nonstandard dynamical systems

If $D = (X,\pi,R)$ is a dynamical system then in an enlargement $*D = (*X,*\pi,*R)$ we see that the flow can be followed for infinite time. This leads to simple characterizations of such standard notions as limit set, stability, prolongations, etc. as well as some notions in the enlargement which have no standard counterpart. At the same time $X$ has been blown up to $*X$ which is a sort of compactification. This leads to effective ways of studying flows on non-compact phase spaces.

Using these ideas we can give necessary and sufficient conditions for topological equivalence of compact metric dynamical systems (Amer. J. Math. 93 (1971), 742-52). Results on almost periodicity have also been obtained.

Albert E. Hurd

Near periods and Bohr compactifications 106

A nonstandard construction is given for the Bohr compactification of a class of uniformly closed, translation invariant subalgebras of the almost periodic functions on a group $G$ which includes the almost periodic functions themselves. The compactification is obtained by

putting an appropriate "weak" topology on the quotient group obtained from *G by factoring out the subgroup of common near periods of all the functions in the subalgebra. A crucial lemma shows that intersections of finitely many sets of translation elements are relatively dense. (June 20, 1973)

H. Jerome Keisler

Monotone complete fields 113

Let F be an ordered field. F is said to be monotone complete if every bounded increasing function $h: F \to F$ has a limit $\lim_{x\to\infty} h(x)$ in F. The cofinality of F is the least cardinal of an unbounded subset of F. The field R of real numbers is the unique monotone complete field of cofinality $\omega$. In nonstandard analysis one uses a proper elementary extension *R of R. *R can never be complete; in this paper we show that *R can be taken to be monotone complete. (Jan. 16, 1973)

Peter J. Kelemen

Quantum mechanics, quantum field theory, hyper-quantum mechanics 116

Applications of nonstandard analysis to quantum physics are considered. Quantum field theory, the theory of systems with countably infinite degrees of freedom is constructed as an external intermediary between quantum mechanics of finite degrees of freedom and its nonstandard extension, hyper-quantum mechanics of star-finite degrees of freedom. The possibility that hyper-quantum mechanics, perhaps modified with external assumptions will provide the foundations for a theory of elementary particles is also discussed. (January 11, 1973)

reflexivity of normed spaces follow. In particular, a very simple and direct nonstandard proof of the theorem that every uniformly convex Banach space is reflexive, exists.

The existence of liftings is usually obtained from the existence of a lower density in the sense of Caratheodory. Extending a result of B. Eifrig (Ein nicht-standard Beweis für die Existenz eines starken Liftings, Archiv der Math. 23 (1972), 425-427) it is shown by using nonstandard methods that given an algebra of sets and a Boolean equivalence relation which admits a lower density, then it admits already a lifting.

Louis Narens

Field embeddings of generalized metric spaces 155

A generalized metric space $X$ is a space whose topology is "metrizable" by a "distance" function into an ordered abelian group $(G,<)$. $(G,<)$ is said to be of type $\Omega$ if and only if $\Omega$ is the smallest cardinality of a subset $Y$ of positive elements of $G$ such that for each positive element of $g$ of $G$ there is a $y \in Y$ such that $y \leq g$. $X$ is said to be of type $\Omega$ if its topology is metrizable by an ordered abelian group $(G,<)$ of type $\Omega$. It is shown that if $X$ is of type $\Omega$ and the cardinality of $X$ is at most $\Omega$, then there is a nonstandard model of the rational numbers $Q$ with the order topology such that $X$ is homeomorphic to a subset of $Q$. Furthermore, if the generalized continuum hypothesis is assumed, it is shown that for each regular cardinal $\Omega$ there is a nonstandard model of the rational numbers $Q$ of cardinality $\Omega$ with the order topology such that each generalized metric space $X$ of type $\Omega$ that has cardinality at most $\Omega$ is homeomorphic to a subset of $Q$. (April 19, 1973)

Louis Narens

Homeomorphism types of generalized metric spaces 171

It is shown that if $X$ and $Y$ are generalized metric spaces of type $\Omega$, $X$ and $Y$ have cardinality $\Omega$, and $X$ and $Y$ have no isolated points, then $X$ and $Y$ are homeomorphic. (April 19, 1973)

Theorem. If $(f_1',\ldots,f_m')$ belongs to the module generated by $(g_{j1}',\ldots,g_{jm}')$, $j = 1,\ldots,k$ then there exist monadic sections $q_j$ at $z$, $j = 1,\ldots,k$ such that $(f_1,\ldots,f_m) = \sum_j q_j(g_{j1},\ldots,g_{jm})$ on the entire monad of $z$.

Corollary. If the $f_j$ are finite on the monad, the $q_j$ also may be chosen so as to be finite on the monad. (December 1, 1972)

K. D. Stroyan

A nonstandard characterization of mixed topologies 261

In this paper we generalize the characterization of the strict and Mackey topologies (on $H^\infty$ and $L^\infty$ respectively) to the general setting of mixed topologies. (December 3, 1973)

K. D. Stroyan

Infinitesimal relations on the space of bounded holomorphic functions

Let U denote the unit disk. On $*(H^\infty(U))$ there are several interesting notions of infinitesimals, for example, pointwise, compact or uniform convergence can be characterized by appropriate infinitesimal relations.

Two other interesting infinitesimal relations which correspond in standard terms to mixed topologies are as follows: h and k in $*(H^\infty(U))$ are within a bounded infinitesimal provided $h(z) \sim k(z)$ for all $z \in *U$ and $h(z) \approx k(z)$ for near standard z; h and k are within a measure infinitesimal provided $h(z) \sim k(z)$ for all z and $h(z) \approx k(z)$ (as radial limits) for nearly all $|z| = 1$ (that is, except on a set of infinitesimal $d\theta$-measure). These correspond to bounded pointwise convergence and bounded $L^1$-convergence.

James K. Thurber and Jose Katz

Applications of fractional powers of delta functions 272

In calculations in electromagnetic theory, quantum scattering theory and perturbation expansions in quantum field theory, it is often desirable for such expressions as

$$\Delta\ \delta(x)\delta(x) \text{ and}$$

$$\int_{-\infty}^{\infty} \delta(x)\delta(x)\theta(x)\theta(-x)dx$$

to have a meaning, where $\delta(x)$ is the Dirac delta function and

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Also sometimes it is important to distinguish between different delta functions such as

$$\lim_{\varepsilon \to +\infty} \frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \qquad \text{and}$$

$$\lim_{n \to +\infty} \left[\frac{n}{\pi}\right]^{1/2} e^{-nx^2}$$

And as we show in this paper it is useful to have fractional powers of the Dirac function, such as

$$\left[\delta(x)\right]^{1/2} \quad \text{and} \quad \left[\delta(x)\right]^{5/6} .$$

If the delta function is conceived solely in terms of the L. Schwarz theory of distributions, all of the above notions lead to serious difficulties and in some instances to unresolvable inconsistencies.

However A. Robinson's theory of Nonstandard Analysis can be used to make perfectly rigorous and useful sense of the above notions. We apply here nonstandard versions of delta functions to some basic calculations in classical electro-magnetic theory, quantum scattering theory, and the star-finite infinitely many body Schrodinger Equation.

In classical electromagnetic theory we are able to give a model of a point electron with a finite self-energy which is consistent with causality requirements. This leads to an interaction between charged particles of infinitesimal strength. In the infinite-star finite body problem this leads however to a finite effective interaction.

The fractional powers of delta functions as used here can be interpreted as a form of renormalization. (December 10, 1972)

Frank Wattenberg

Two topologies with the same monads 303

Suppose $X$ is a $\sigma$-compact, locally compact space and consider the space $F_C(X)$ of continuous functions $X \to R$ with compact support. Two of the most interesting topologies on this space

are the direct limit (or inductive) topology and the $\delta$-topology. These two distinct topologies have remarkably similar properties. This paper investigates these two topologies using the techniques of Nonstandard Analysis. The main result of this paper is that if $D$ is an ultrafilter on $\omega$ which is a P-point and ${}^*F_C(X)$ is the ultrapower of $F_C(X)$ with respect to $D$ then the direct limit and $\delta$-topologies have the same monads. Thus, from the point of view of Nonstandard Analysis it is clear why the two topologies share so many properties. (January 25, 1973)

Elias Zakon

A new variant of nonstandard analysis 313

In this paper, we generalize the concept of a monomorphism of one superstructure into another, as previously introduced by A. Robinson and E. Zakon ("A set theoretical characterization of enlargements, [8]"). As a novelty, we consider "non-strict" monomorphisms under which an internal set may possess external elements. This generalization of the ordinary "strict" monomorphisms (including Robinson's "enlargements" as a special case), admits a particularly simple definition. One of our aims is to demonstrate that "strictness" is not needed for the validity of many important theorems. We give a self-contained presentation of all basic notions of Nonstandard Analysis as they evolve from the newly defined "monomorphisms". (February 10, 1973)