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Andrew Adler

An application of elementary model theory to topological Boolean algebras

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Every topological Boolean algebra is isomorphic to a topological subalgebra of the ordered space on a non-standard model of the rationals. So in particular a countable topological boolean algebra is isomorphic to a topological subalgebra of the ordered space on an  $\eta_1$  set. An atomic countable  $T_3$  dense-in-itself algebra is isomorphic to a topological subalgebra of the space of rationals. (January 10, 1973)

Michael Behrens

Analytic sets in  $\mathfrak{M}(D)$

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Nonstandard techniques are used to investigate the maximal ideal space  $\mathfrak{M}(D)$  of the Banach algebra  $H^\infty(D)$  of bounded analytic functions on a planar domain  $D$ , and, more especially, to study the analytic structure in  $\mathfrak{M}(D) - D$ . A relatively complete discussion is presented for the unit disk, and a few results for infinitely connected domains are discussed. (August 1, 1973)

Michael Behrens

Boundary values for meromorphic functions defined in the open unit disk

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Several classical boundary value theorems including the Gross principal value theorem are proved using nonstandard methods. (August 20, 1973)

Michael Behrens

A local inverse function theorem

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An inverse function theorem is proved under a local assumption which is weaker than differentiability in a neighborhood. (August 20, 1973)

Steven F. Bellenot

Nonstandard topological vector spaces

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A simple short nonstandard proof of the uniqueness of finite dimensional Hausdorff topological vector spaces. (October 17, 1972)

Allen R. Bernstein and Peter A. Loeb

A nonstandard integration theory for unbounded functions

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This paper applies the nonstandard partition method of measure theory to the problem of integrating unbounded functions in a linear fashion, i.e., without truncation. As an application, one obtains a projection  $T$  of the extension  $*L_1(X, \mathcal{M}, \mu)$  of an arbitrary  $L_1$  space onto a  $*$  finite dimensional subset of  $*L_1(X, \mathcal{M}, \mu)$  so that if  $h \in L_1(X, \mathcal{M}, \mu)$  then  $\|T^*h - *h\| \approx 0$ . (March 5, 1973)

Allen R. Bernstein and Frank Wattenberg

Cardinality-dependent properties of topological spaces

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In this paper we characterize certain topological properties whose definitions depend on a particular infinite cardinal by using ultrapowers over sets of that cardinality. In particular we obtain a nonstandard characterization of  $\alpha$ -Baire spaces in terms of certain  $\alpha$ -indexed ultrapowers. As a corollary, Baire spaces have a nonstandard characterization valid in any non-trivial countably-indexed ultrapower. This characterization may be used to give nonstandard proofs of results which depend on the Baire category theorem. This provides, at least in part, a solution to problem number 9 asked by Abraham Robinson in his retiring presidential address delivered to the Association for Symbolic Logic, January 1973 in Dallas, Texas. The paper also uses countably-indexed ultrapowers to examine certain countable equivalence conditions between topological spaces. (April 20, 1973)

Donald J. Brown and Abraham Robinson

Nonstandard exchange economies

An exchange economy consists of a set of traders each of whom is characterized by an initial endowment and a preference relation. In addition, one usually assumes that the set of traders is finite. But

in order to model perfectly competitive markets, i.e., markets where each trader's economic influence is negligible, we assume that the economy has  $\omega$  traders, where  $\omega$  is an infinite integer, and the average endowment of each trader is infinitesimal.

In these nonstandard exchange economies, we examine the relationship between outcomes obtained through bargaining, called core allocations, and the allocations arising out of the competitive price system. We show that Edgeworth's conjecture, that every core allocation is a competitive allocation, is true in nonstandard exchange economies. As a consequence of this theorem we also show that core allocations in large finite standard economies are approximately competitive allocations.

References: Brown, D. J. and A. Robinson, "Nonstandard Exchange Economies", Econometrica (to appear).

———, "A Limit Theorem on the Cores of Large Standard Exchange Economies," Proc.Nat.Acad.Sc., U.S.A., Vol.69, No. 5, 1258-1260.

Harry Gonshor

Enlargements contain various kinds of completions

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In this paper we show that various types of completions in different senses may be obtained as subquotients of enlargements. Among the examples considered are the following: The rationals as a "completion" of the integers, the Stone-Cech compactification of a completely regular space, the second conjugate space of a Banach space, rings of quotients of rings of continuous functions, the projective cover of a compact Hausdorff space, and the completions of a Boolean algebra. The last example is studied in detail. (March 22, 1973)

C. Ward Henson and L. C. Moore, Jr.

Semi-reflexivity of the nonstandard hulls of a locally convex space

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Let  $E, F$  be vector spaces over  $R$  or  $C$  paired by a nonsingular bilinear form  $\langle \dots, \dots \rangle$  and let  $\theta$  be a locally convex topology on  $E$  which is admissible for the pairing. Given an enlargement  ${}^*\mathfrak{M}$  of a set theoretical structure  $\mathfrak{M}$  which contains  $E$  and  $F$ , let  $(\hat{E}, \hat{\theta})$  be the associated nonstandard hull of  $(E, \theta)$ . Also let  $\hat{F}$  be the space of linear functionals on  $\hat{E}$  which are represented by those points  $q$  in  ${}^*F$  for which  $\langle p, q \rangle$  is finite for all  $\theta$ -finite  $p$  in  ${}^*E$ . Then  $\hat{F}$  is contained in the dual space  $\hat{E}'$  of  $(\hat{E}, \hat{\theta})$  and  $\hat{\theta}$  is an admissible topology relative to the pairing between  $\hat{E}$  and  $\hat{F}$ .

The principal result is that  $(\hat{E}, \hat{\theta})$  is semi-reflexive if and only if  $\hat{F}$  is  $\beta(\hat{E}', \hat{E})$ -dense in  $\hat{E}'$ . (This extends a result for normed spaces proved earlier by the authors.) Moreover, a geometric condition on  $(E, \theta)$  is given which is equivalent to the semi-reflexivity of  $(\hat{E}, \hat{\theta})$  (and therefore this property of  $(\hat{E}, \hat{\theta})$  does not depend on the particular enlargement  $*\mathfrak{m}$ ). The main technical tool is the following result which seems of interest itself.

Retraction Theorem: If  $*\mathfrak{m}$  is  $\kappa$ -saturated and  $S$  is a subspace of  $\hat{E}$  which has Hamel dimension less than  $\kappa$ , then for each  $\phi \in \hat{E}'$  there exists  $\psi \in \hat{F}$  such that  $\phi(x) = \psi(x)$  for all  $x \in S$ . (March 8, 1973)

C. Ward Henson and L. C. Moore, Jr.

Invariance of the nonstandard hulls of a uniform space

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Let  $(X, \mathfrak{u})$  be a uniform space and let  $*\mathfrak{m}$  be an enlargement of some set-theoretical structure  $\mathfrak{m}$  which contain  $X$ . It is shown that if  $p \in *X$  is not  $\mathfrak{u}$ -pre-nearstandard and if  $*\mathfrak{m}$  is  $\kappa$ -saturated, then the filter monad  $\mu(\text{Fil}(p)) = \cap \{ *Y \mid p \in *Y \}$  intersects at least  $\kappa$  distinct  $\mathfrak{u}$ -monads  $\mu_{\mathfrak{u}}(q)$ . As a consequence, either (i) the nonstandard hulls of  $(X, \mathfrak{u})$  (constructed using the  $\mathfrak{u}$ -finite elements of  $*X$ ) are all equal to the completion of  $(X, \mathfrak{u})$  and are therefore independent of  $*\mathfrak{m}$  or (ii) the nonstandard hulls of  $(X, \mathfrak{u})$  can be made arbitrarily large by varying the choice of  $*\mathfrak{m}$ . Similar results hold for the classes of nonstandard hulls defined by Luxemburg. In case (i) holds, we say  $(X, \mathfrak{u})$  has invariant nonstandard hulls (extending the terminology introduced by the authors for locally convex topological vector spaces).

A metric space  $(X, d)$  has invariant nonstandard hulls if and only if every subset of  $X$  on which every uniformly continuous function is bounded must be totally bounded. This equivalence is not true, however, for arbitrary uniform spaces. It is also shown that if  $(X, \mathfrak{T})$  is a separable, topologically complete (metrizable) space, then  $\mathfrak{T}$  can be defined by a metric  $d$  such that  $(X, d)$  is complete and has invariant nonstandard hulls. (March 8, 1973)

S. Feferman, D. Scott, and S. Tennenbaum proved that every non-trivial homomorphic image of the semi-ring  $R$  of recursive functions fails to be a model of arithmetic. The aim of this paper is to show that every countable model of full arithmetic can be embedded in such a homomorphic image. To prove this we modify a theorem by H. Friedman to obtain a sufficient condition for a model of arithmetic to be embeddable in a model of a fragment of arithmetic. We then introduce recursive ultrapowers - homomorphic images of  $R$  which are models of that fragment. Finally, given a model, we show how to construct a recursive ultrapower which satisfies the condition of Friedman's theorem. (April 3, 1973.)

Albert E. Hurd

#### Nonstandard dynamical systems

If  $D = (X, \pi, R)$  is a dynamical system then in an enlargement  $*D = (*X, *\pi, *R)$  we see that the flow can be followed for infinite time. This leads to simple characterizations of such standard notions as limit set, stability, prolongations, etc. as well as some notions in the enlargement which have no standard counterpart. At the same time  $X$  has been blown up to  $*X$  which is a sort of compactification. This leads to effective ways of studying flows on non-compact phase spaces.

Using these ideas we can give necessary and sufficient conditions for topological equivalence of compact metric dynamical systems (Amer. J. Math. 93 (1971), 742-52). Results on almost periodicity have also been obtained.

Albert E. Hurd

#### Near periods and Bohr compactifications

A nonstandard construction is given for the Bohr compactification of a class of uniformly closed, translation invariant subalgebras of the almost periodic functions on a group  $G$  which includes the almost periodic functions themselves. The compactification is obtained by

putting an appropriate "weak" topology on the quotient group obtained from  $*G$  by factoring out the subgroup of common near periods of all the functions in the subalgebra. A crucial lemma shows that intersections of finitely many sets of translation elements are relatively dense. (June 20, 1973)

H. Jerome Keisler

Monotone complete fields

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Let  $F$  be an ordered field.  $F$  is said to be monotone complete if every bounded increasing function  $h: F \rightarrow F$  has a limit  $\lim_{x \rightarrow \infty} h(x)$  in  $F$ . The cofinality of  $F$  is the least cardinal of an unbounded subset of  $F$ . The field  $R$  of real numbers is the unique monotone complete field of cofinality  $\omega$ . In nonstandard analysis one uses a proper elementary extension  $*R$  of  $R$ .  $*R$  can never be complete; in this paper we show that  $*R$  can be taken to be monotone complete. (Jan. 16, 1973)

Peter J. Kelemen

Quantum mechanics, quantum field theory, hyper-quantum mechanics

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Applications of nonstandard analysis to quantum physics are considered. Quantum field theory, the theory of systems with countably infinite degrees of freedom is constructed as an external intermediary between quantum mechanics of finite degrees of freedom and its nonstandard extension, hyper-quantum mechanics of star-finite degrees of freedom. The possibility that hyper-quantum mechanics, perhaps modified with external assumptions will provide the foundations for a theory of elementary particles is also discussed. (January 11, 1973)

Anders Kock and Chr. Juul Mikkelsen

Topos-theoretic factorization of nonstandard extensions

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An organization in terms of topos theory and closed category theory of the change in the higher order logic of nonstandard arguments taking place by means of the word 'internal'. Formally, we prove a factorization theorem for first-order logic preserving functors between toposes. Such a functor can be factored into two such functors, where the first preserves higher order logic and the second preserves elements.

(January 23, 1973)

Peter A. Loeb

A nonstandard representation of Borel measures and  $\sigma$ -finite measures

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It is shown that any extension of the real line contains a linearly ordered  $\sigma$ -finite collection of compact sets on which every finite Borel measure is essentially concentrated. A nonstandard proof of Lusin's theorem is given, and a method of transforming  $\sigma$ -finite measures into counting measures is established using partitions of the measure space. (March 5, 1973)

Peter A. Loeb

A note on continuity for Robinson's predistributions

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It is shown that a predistribution which is standard at each standard point has a continuous standard part. (March 5, 1973)

W.A.J. Luxemburg

On a theorem of Helly and a theorem about liftings

Helly's theorem in the theory of normed spaces concerning the existence of a solution of a system of linear equations is essentially a statement of the type that a certain binary relation is concurrent. This fact can be used to show that the second dual space of a given normed linear space can be imbedded in the nonstandard hull of the given normed space. From this fact the various theorems concerning

reflexivity of normed spaces follow. In particular, a very simple and direct nonstandard proof of the theorem that every uniformly convex Banach space is reflexive, exists.

The existence of liftings is usually obtained from the existence of a lower density in the sense of Caratheodory. Extending a result of B. Eifrig (Ein nicht-standard Beweis für die Existenz eines starken Liftings, Archiv der Math. 23 (1972), 425-427) it is shown by using nonstandard methods that given an algebra of sets and a Boolean equivalence relation which admits a lower density, then it admits already a lifting.

Louis Narens

#### Field embeddings of generalized metric spaces

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A generalized metric space  $X$  is a space whose topology is "metrizable" by a "distance" function into an ordered abelian group  $(G, <)$ .  $(G, <)$  is said to be of type  $\Omega$  if and only if  $\Omega$  is the smallest cardinality of a subset  $Y$  of positive elements of  $G$  such that for each positive element  $g$  of  $G$  there is a  $y \in Y$  such that  $y \leq g$ .  $X$  is said to be of type  $\Omega$  if its topology is metrizable by an ordered abelian group  $(G, <)$  of type  $\Omega$ . It is shown that if  $X$  is of type  $\Omega$  and the cardinality of  $X$  is at most  $\Omega$ , then there is a nonstandard model of the rational numbers  $\mathbb{Q}$  with the order topology such that  $X$  is homeomorphic to a subset of  $\mathbb{Q}$ . Furthermore, if the generalized continuum hypothesis is assumed, it is shown that for each regular cardinal  $\Omega$  there is a nonstandard model of the rational numbers  $\mathbb{Q}$  of cardinality  $\Omega$  with the order topology such that each generalized metric space  $X$  of type  $\Omega$  that has cardinality at most  $\Omega$  is homeomorphic to a subset of  $\mathbb{Q}$ . (April 19, 1973)

Louis Narens

#### Homeomorphism types of generalized metric spaces

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It is shown that if  $X$  and  $Y$  are generalized metric spaces of type  $\Omega$ ,  $X$  and  $Y$  have cardinality  $\Omega$ , and  $X$  and  $Y$  have no isolated points, then  $X$  and  $Y$  are homeomorphic. (April 19, 1973)



Rohit Parikh & Milton Parnes

## Conditional probabilities and uniform sets

180

We show how to give a nonstandard construction for conditional probability functions on the real line which satisfy strong regularity conditions (these are not unique). Applications are given to define the notion of uniformly distributed set, and to the theory of determinate pairs of sets  $A, B$  where the conditional probability does not depend on the particular probability function chosen. (April 16, 1973)

R. G. Phillips

## Omitting types in arithmetic and conservative extensions

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In this paper we show that each model  $M$  of Peano's Axioms has a proper elementary extension, called a conservative extension, which defines no new relations on  $M$ . We show that conservative extensions have a canonical additive group which is minimal in structure. Finally we compare conservative extensions with Gaifman's minimal extensions. (February 10, 1973)

David Pincus

## The strength of the Hahn-Banach theorem

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The Hahn-Banach theorem is proved to be strictly weaker than the prime ideal theorem for Boolean algebras. A "standard" proof of this for set theory with atoms appears in Bull. A.M.S. 78 (1972), 766-770. A proof for full Zermelo Fraenkel set theory uses nonstandard analysis. There are also some results on statements which are weaker than the Hahn-Banach theorem. (April 20, 1973)

Abraham Robinson

## Enlarged sheaves

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Enlarged sheaves are considered, particularly for sheaves of holomorphic functions of several complex variables. A monadic section is the restriction of a section to a monad in its domain. Let  $g_{ji}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, k$  be standard monadic sections at a standard point  $z$ , and let  $g'_{ji}$  be the corresponding germs at  $z$ . Let  $f_j$ ,  $j = 1, \dots, m$  be internal monadic sections with germs  $f'_j$  at  $z$ .

Theorem. If  $(f'_1, \dots, f'_m)$  belongs to the module generated by  $(g'_{j1}, \dots, g'_{jm})$ ,  $j = 1, \dots, k$  then there exist monadic sections  $q_j$  at  $z$ ,  $j = 1, \dots, k$  such that  $(f_1, \dots, f_m) = \sum_j q_j (g_{j1}, \dots, g_{jm})$  on the entire monad of  $z$ .

Corollary. If the  $f_j$  are finite on the monad, the  $q_j$  also may be chosen so as to be finite on the monad. (December 1, 1972)

K. D. Stroyan

A nonstandard characterization of mixed topologies

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In this paper we generalize the characterization of the strict and Mackey topologies (on  $H^\infty$  and  $L^\infty$  respectively) to the general setting of mixed topologies. (December 3, 1973)

K. D. Stroyan

Infinitesimal relations on the space of bounded holomorphic functions

Let  $U$  denote the unit disk. On  $*(H^\infty(U))$  there are several interesting notions of infinitesimals, for example, pointwise, compact or uniform convergence can be characterized by appropriate infinitesimal relations.

Two other interesting infinitesimal relations which correspond in standard terms to mixed topologies are as follows:  $h$  and  $k$  in  $*(H^\infty(U))$  are within a bounded infinitesimal provided  $h(z) \sim k(z)$  for all  $z \in *U$  and  $h(z) \approx k(z)$  for near standard  $z$ ;  $h$  and  $k$  are within a measure infinitesimal provided  $h(z) \sim k(z)$  for all  $z$  and  $h(z) \approx k(z)$  (as radial limits) for nearly all  $|z| = 1$  (that is, except on a set of infinitesimal  $d\theta$ -measure). These correspond to bounded pointwise convergence and bounded  $L^1$ -convergence.

James K. Thurber and Jose Katz

Applications of fractional powers of delta functions

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In calculations in electromagnetic theory, quantum scattering theory and perturbation expansions in quantum field theory, it is often desirable for such expressions as

$$\Delta \delta(x) \delta(x) \text{ and } \int_{-\infty}^{\infty} \delta(x) \delta(x) \theta(x) \theta(-x) dx$$

to have a meaning, where  $\delta(x)$  is the Dirac delta function and

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Also sometimes it is important to distinguish between different delta functions such as

$$\lim_{\epsilon \rightarrow +\infty} \frac{\epsilon}{\pi(x^2 + \epsilon^2)} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \left[ \frac{n}{\pi} \right]^{1/2} e^{-nx^2}$$

And as we show in this paper it is useful to have fractional powers of the Dirac function, such as

$$[\delta(x)]^{1/2} \quad \text{and} \quad [\delta(x)]^{5/6}.$$

If the delta function is conceived solely in terms of the L. Schwarz theory of distributions, all of the above notions lead to serious difficulties and in some instances to unresolvable inconsistencies.

However A. Robinson's theory of Nonstandard Analysis can be used to make perfectly rigorous and useful sense of the above notions. We apply here nonstandard versions of delta functions to some basic calculations in classical electro-magnetic theory, quantum scattering theory, and the star-finite infinitely many body Schrodinger Equation.

In classical electromagnetic theory we are able to give a model of a point electron with a finite self-energy which is consistent with causality requirements. This leads to an interaction between charged particles of infinitesimal strength. In the infinite-star finite body problem this leads however to a finite effective interaction.

The fractional powers of delta functions as used here can be interpreted as a form of renormalization. (December 10, 1972)

Frank Wattenberg

Two topologies with the same monads

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Suppose  $X$  is a  $\sigma$ -compact, locally compact space and consider the space  $F_C(X)$  of continuous functions  $X \rightarrow \mathbb{R}$  with compact support. Two of the most interesting topologies on this space

are the direct limit (or inductive) topology and the  $\delta$ -topology. These two distinct topologies have remarkably similar properties. This paper investigates these two topologies using the techniques of Nonstandard Analysis. The main result of this paper is that if  $D$  is an ultrafilter on  $\omega$  which is a P-point and  ${}^*F_C(X)$  is the ultrapower of  $F_C(X)$  with respect to  $D$  then the direct limit and  $\delta$ -topologies have the same monads. Thus, from the point of view of Nonstandard Analysis it is clear why the two topologies share so many properties. (January 25, 1973)

Elias Zakon

A new variant of nonstandard analysis

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In this paper, we generalize the concept of a monomorphism of one superstructure into another, as previously introduced by A. Robinson and E. Zakon ("A set theoretical characterization of enlargements, [8]"). As a novelty, we consider "non-strict" monomorphisms under which an internal set may possess external elements. This generalization of the ordinary "strict" monomorphisms (including Robinson's "enlargements" as a special case), admits a particularly simple definition. One of our aims is to demonstrate that "strictness" is not needed for the validity of many important theorems. We give a self-contained presentation of all basic notions of Nonstandard Analysis as they evolve from the newly defined "monomorphisms". (February 10, 1973)