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Dynamical Systems

An Introduction with Applications
in Economics and Biology

Second Revised
and Enlarged Edition

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