

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	What is Wave Turbulence?	1
1.2	Historic Remarks	1
1.3	Recent Developments	2
1.3.1	Rapid Expansion of WT Applications	3
1.3.2	Highly Improved Quality of Experimental Data and Numerical Simulations of WT Systems	4
1.3.3	Discovery of Importance of Coherent Structures in WT Evolution	5
1.3.4	Theory Extension Beyond Spectra	6
1.3.5	Study of the Finite-Box Effects	6
1.4	What is this Book About?	7
	References	9

## Part I Primer on Wave Turbulence

<b>2</b>	<b>Wave Turbulence as a Part of General Turbulence Theory</b>	<b>17</b>
2.1	Basic Facts about Hydrodynamic Turbulence	17
2.1.1	Richardson Cascade	18
2.1.2	Kolmogorov–Obukhov Theory	19
2.1.3	2D Turbulence	20
2.2	Placing Wave Turbulence in the Context of General Turbulence	25
2.2.1	Common Turbulence Properties	25
2.2.2	Distinct Properties of WT	26
	References	27
<b>3</b>	<b>For the Impatient: A WT Cheatsheet</b>	<b>29</b>
3.1	Weak Wave Turbulence	30

3.1.1	Three-Wave? Four-Wave? N-Wave? . . . . .	30
3.1.2	Dimensional Derivation of KZ Spectra . . . . .	32
3.1.3	Examples . . . . .	34
3.2	Strong Wave Turbulence and Critical Balance . . . . .	40
3.2.1	MHD Turbulence . . . . .	40
3.2.2	Gravity Water Waves . . . . .	41
3.2.3	Stratified Turbulence . . . . .	42
3.2.4	Rotating Turbulence . . . . .	43
3.2.5	Quasi-Geostrophic Turbulence . . . . .	44
3.2.6	Kelvin Waves . . . . .	45
3.2.7	Burgers? KdV? . . . . .	46
	References . . . . .	47
<b>4</b>	<b>Solutions to Exercises . . . . .</b>	<b>49</b>
4.1	Fjørtoft Argument in Terms of Centroids: Exercise 2.1 . . . . .	49
4.2	$k$ -Centroids versus $l$ -centroids: Exercise 2.2 . . . . .	50
4.3	Four-Wave Resonances in 1D Systems: Exercise 3.1 . . . . .	50
4.4	Four-Wave $3 \rightarrow 1$ Resonances in 2D Systems: Exercise 3.2. . . . .	51
	References . . . . .	52

## Part II Wave Turbulence Closures

<b>5</b>	<b>Statistical Objects in Wave Turbulence . . . . .</b>	<b>55</b>
5.1	Statistical Variables. . . . .	55
5.2	Probability Density Functions. . . . .	57
5.3	Random Phase and Amplitude Fields . . . . .	58
5.4	Generating Functions. . . . .	59
5.5	Wave Spectrum, Higher Moments and Structure Functions . . . . .	60
5.6	RPA Averaging . . . . .	63
	References . . . . .	66
<b>6</b>	<b>Wave Turbulence Formalism . . . . .</b>	<b>67</b>
6.1	Dharmachakra of Wave Turbulence: Main Steps, Ideas and Building Blocks . . . . .	67
6.1.1	Mahayana (Comprehensive Scheme) . . . . .	68
6.1.2	Hinayana (Reduced Scheme) . . . . .	71
6.2	Master Example: Petviashvili Equation. . . . .	71
6.2.1	Conservation Laws . . . . .	72
6.2.2	Fourier Space . . . . .	73
6.2.3	Interaction Representation . . . . .	74
6.3	Weak Nonlinearity Expansion . . . . .	75
6.3.1	Solution for the Wave Amplitudes at Intermediate Times . . . . .	75

6.3.2	Weak Nonlinearity Expansion for the Generating Function . . . . .	76
6.4	Statistical Averaging . . . . .	77
6.5	Large-Box and Weak-Nonlinearity Limits . . . . .	80
6.5.1	Taking $L \rightarrow \infty$ . . . . .	81
6.5.2	Taking $\epsilon \rightarrow 0$ . . . . .	81
6.6	The PDF . . . . .	83
6.7	Kinetic Equation . . . . .	84
6.7.1	Symmetrical Form of the Kinetic Equation . . . . .	85
6.8	Generalization to Complex Wavefields . . . . .	86
6.8.1	Hamiltonian Wave Equations . . . . .	86
6.9	Four-Wave and Higher-Order Systems . . . . .	88
6.9.1	Four-Wave Systems . . . . .	88
6.9.2	Systems with Higher-Order Wave Resonances . . . . .	91
6.10	Evolution of Multi-Mode Statistics . . . . .	94
6.10.1	Weak Nonlinearity Expansion of the Generating Function . . . . .	95
6.10.2	Statistical Averaging and Graphs . . . . .	96
6.10.3	Equation for $\mathcal{Z}^{(N)}$ . . . . .	100
6.10.4	Equation for the PDF . . . . .	101
6.11	Generalization to the Four-Wave and the Higher-Order Systems . . . . .	103
	References . . . . .	104
<b>7</b>	<b>Solutions to Exercises . . . . .</b>	<b>107</b>
7.1	One-Mode Generating Function for Gaussian Fields: Exercise 5.1 . . . . .	107
7.2	One-Mode Moments for Gaussian Fields: Exercise 5.2 . . . . .	107
7.3	Six-Order Multi-Point Moment: Exercise 5.3 . . . . .	108
7.4	Fourth-Order Structure Function: Exercise 5.4 . . . . .	109
7.5	Invariants of the Petviashvili Equation: Exercise 6.1 . . . . .	109
7.6	Charney-Hasegawa-Mima model: Exercise 6.2 . . . . .	110
7.7	$T \rightarrow \infty$ Limit: Exercise 6.4 . . . . .	111
7.8	Slow and Fast Timescales in the Wave Amplitude Evolution: Exercise 6.7 . . . . .	112
7.9	Eliminating $U_{123}$ : Exercise 6.9 . . . . .	113
7.10	Nonlinear Phase Evolution: Exercise 6.11 . . . . .	113
7.11	Inconsistency of WT Expansions Without Frequency Re-Normalization: Exercise 6.12 . . . . .	114
7.12	Finding $G_3$ – $G_5$ : Exercise 6.16 . . . . .	115
7.13	Appendix: Interaction Coefficient for the Deep Water Surface Waves . . . . .	115
	References . . . . .	116

### Part III Wave Turbulence Predictions

<b>8</b>	<b>Conserved Quantities in Wave Turbulence and their Cascades . . .</b>	<b>119</b>
8.1	Conserved Quantities in Wave Turbulence. . . . .	119
8.1.1	Energy and Momentum . . . . .	119
8.1.2	Three-Wave Systems. . . . .	120
8.1.3	Four-Wave Systems. . . . .	122
8.1.4	Conservation Laws in the Multi-Particle Statistics . . . . .	123
8.1.5	Relation Between the Dynamical and the Statistical Invariants . . . . .	125
8.2	Directions of Turbulent Cascades . . . . .	126
8.2.1	Dual Cascade in the NLS and Other Even-Wave Systems . . . . .	126
8.2.2	Cascade of Momentum and Other Non-Positive Invariants . . . . .	128
8.2.3	Triple Cascade in the Petviashvili and Other Rossby/Drift Wave Systems . . . . .	129
	References . . . . .	132
<b>9</b>	<b>Steady State and Evolving Solutions for the Wave Spectrum . . .</b>	<b>133</b>
9.1	Thermodynamic Equilibrium States: Rayleigh-Jeans Spectra . . . . .	133
9.2	Cascade States: Kolmogorov-Zakharov Spectra . . . . .	134
9.2.1	Three-Wave Systems. . . . .	135
9.2.2	Four-Wave Systems. . . . .	140
9.2.3	Temporal Evolution Leading to KZ Spectra: Finite and Infinite Capacity Systems . . . . .	143
9.2.4	KZ Spectra in Anisotropic Media . . . . .	146
9.2.5	Other Power-Law Spectra in Anisotropic Media . . .	148
9.2.6	Locality and Stability . . . . .	150
	References . . . . .	161
<b>10</b>	<b>Finite-Size Effects in Wave Turbulence. . . . .</b>	<b>163</b>
10.1	Small-Box Regime: Discrete Turbulence . . . . .	164
10.2	Infinite-Box Regime: Kinetic Wave Turbulence . . . . .	166
10.3	Mesoscopic Turbulence: Sandpile Behavior . . . . .	167
10.4	Coexistence of Different Regimes in the $\mathbf{k}$ -Space . . . . .	169
10.5	Cascade Tree in the Discrete $\mathbf{k}$ -Space . . . . .	169
	References . . . . .	171

<b>11</b>	<b>Properties of the Higher-Order Statistics. Intermittency and WT Life Cycle</b>	173
11.1	Solutions for the One-Mode PDFs and the Moments	173
11.2	Wave Turbulence Life Cycle	177
11.3	Solutions for the $N$ -Mode Joint PDF's	180
11.4	Validity of RPA	182
	References	184
<b>12</b>	<b>Solutions to Exercises</b>	185
12.1	Zonostrophy Invariant: Exercise 8.1	185
12.2	Waveaction Conservation for the Four-Wave Systems: Exercise 8.3	186
12.3	Rayleigh-Jeans Solutions: Exercise 9.1	186
12.4	Energy Flux Direction in Systems with a Single Relevant Dimensional Parameter: Exercise 9.2	187
12.5	Zakharov Transform for the Four-Wave Systems: Exercise 9.3	187
12.6	Geometrical Condition of Stability: Exercise 9.9	187
	Reference	188

## Part IV Selected Applications

<b>13</b>	<b>Nonlocal Drift/RossbyWave Turbulence</b>	191
13.1	When is Turbulence Nonlocal?	191
13.2	Nonlocal weak Drift/Rossby Turbulence	192
	13.2.1 Nonlocal Interaction with Large Scales	192
	13.2.2 Evolution of Nonlocal Rossby/Drift Turbulence: a Feedback Loop	194
	13.2.3 Nonlocal Interaction with Small-Scale Zonal Flows	197
13.3	Beyond Weak Turbulence: Two Regimes of Zonal-Flow Growth	199
	13.3.1 Weak ZF: Diffusive Regime	202
	13.3.2 Strong ZF: Rapid Distortion Regime	203
	13.3.3 Transition Between the Two Regimes of the Zonal Flow Generation	204
13.4	Numerical Modeling of the Forced-Dissipated CHM Equation	204
13.5	Solution to Exercise	207
	13.5.1 Relation Between the Spectrum and the Velocity of the Large Scales: Exercise 13.1	207
	References	207

<b>14</b>	<b>Magneto-Hydrodynamic Turbulence</b>	209
14.1	Introduction	209
14.2	Reduced MHD model	210
14.3	Very Weak WT: Discrete Regime and 2D Enslaving	213
14.4	Large-Box Limit: Kinetic Regime	215
14.4.1	Weak Nonlinearity Expansion	215
14.4.2	Statistical Averaging	216
14.4.3	Conditions of Realizability of the Kinetic Regime	220
14.4.4	Spectra in the Kinetic Regime: Energy Cascades—Balanced and Imbalanced Turbulence	221
14.4.5	Cross-Helicity	223
14.4.6	Transient Evolution Leading to Formation of the KZ Spectrum	224
14.4.7	PDF's in the Kinetic Regime: Turbulence Intermittency	224
14.5	Mesosopic MHD Wave Turbulence	226
14.6	Summary	227
14.7	Further Reading	227
	References	229
<b>15</b>	<b>Bose-Einstein Condensation</b>	231
15.1	Introduction	231
15.2	Kinetic Equation for the Wave Spectrum	231
15.3	Role of Thermodynamic Solutions	232
15.4	Non-Equilibrium Condensation and KZ Spectra	236
15.5	Differential Approximation Model	238
15.5.1	DAM for NLS Wave Turbulence	238
15.5.2	What Happens When a Pure KZ Spectrum Corresponds to “Wrong” Flux Direction?	240
15.5.3	Extending BEC Description to Include Thermal Clouds	242
15.5.4	Wave-Particle Crossover in Turbulent BEC Cascades	244
15.6	Transient Evolution, Self-Similar Spectra	245
15.7	Breakdown of the Weak Four-Wave Turbulence and Transition to a Three-Wave Regime	246
15.7.1	WT on Background of Strong Condensate	246
15.7.2	Strongly Nonlinear Transition Between the Two Weakly Nonlinear Regimes	248
15.8	Direct Cascade in 3D NLS	254
15.9	Inhomogeneous WT in a Trapping Potential	256
15.10	Condensation in 1D Systems: Optical Turbulence	259
15.11	Summary	262

15.12	Solutions to Exercises . . . . .	263
15.12.1	Direct Cascade in 2D NLS: Exercise 15.2 . . . . .	263
15.12.2	Front Solution for Inverse Cascade in 2D NLS: Exercise 15.3 . . . . .	263
15.12.3	KZ Solutions and Flux Directions for Boltzmann Gas: Exercise 15.4 . . . . .	264
15.12.4	Front Solutions for Boltzmann: Exercise 15.5. . . . .	265
15.12.5	Madelung Transformation: Exercise 15.6 . . . . .	265
15.12.6	KZ Spectra for 1D Optical Turbulence: Exercise 15.7 . . . . .	265
15.12.7	DAM for 1D Optical Turbulence: Exercise 15.8 . . .	267
	References . . . . .	267
16	List of Projects. . . . .	269
16.1	Differential Approximation Models for WT and for Strong Turbulence. . . . .	269
16.2	Collapses and Their Role in WT Cycle . . . . .	271
16.3	Modulational Instability and its Role in WT . . . . .	271
16.4	Interacting Particle Systems . . . . .	272
16.5	Superfluid Turbulence . . . . .	273
16.6	Gravity Water Wave Turbulence . . . . .	275
16.7	Metal-Plate Wave Turbulence . . . . .	275
	References . . . . .	276