

Contents

Preface — V

Notations — VII

Introduction — XVII

Part I: Trace and integral formulas

1	Bounded operators and pseudodifferential operators — 3
1.1	Bounded operators on Hilbert space and traces — 3
1.1.1	Singular values and ideals — 4
1.1.2	Traces on $\mathcal{L}_{1,\infty}$ — 7
1.2	Submajorization and interpolation — 12
1.2.1	Fully symmetric ideals — 12
1.2.2	Lorentz function spaces — 15
1.2.3	Noncommutative Lorentz spaces — 20
1.3	Operator inequalities — 24
1.3.1	Submajorization and majorization inequalities — 24
1.3.2	Hölder inequalities and commutators in Lorentz ideals — 31
1.4	Double operator integrals — 41
1.4.1	Integration of operator-valued functions in the weak operator topology — 41
1.4.2	Definition of a double operator integral — 47
1.4.3	Double operator integrals and estimates of commutators — 50
1.5	Compactness estimates of product-convolution operators — 54
1.5.1	Product-convolution operators on the Euclidean plane — 54
1.5.2	Hilbert–Schmidt operators on the Euclidean plane and the trace formula — 57
1.5.3	Compactness of product-convolution operators for square-integrable functions — 60
1.5.4	Abstract product-convolution estimates — 63
1.5.5	Estimates for product-convolution operators on the Euclidean plane — 74
1.5.6	Estimates for product-convolution operators on tori — 78
1.6	Pseudodifferential operators — 81
1.6.1	The Schwartz space and distributions — 82
1.6.2	Symbols and operators — 83

1.6.3	Sobolev spaces and mapping properties of pseudodifferential operators — 87
1.6.4	Inverses and complex powers of pseudodifferential operators — 92
1.6.5	Compactness estimates for pseudodifferential operators — 107
1.6.6	Asymptotic expansion of symbols — 110
1.6.7	The noncommutative residue — 115
1.7	Notes — 118
2	Trace formulas — 129
2.1	Introduction — 129
2.2	The Banach algebra of modulated functions — 132
2.2.1	Closure of the algebra of modulated functions — 133
2.2.2	Square modulated functions and behavior of the Fourier transform — 134
2.3	Laplacian modulated operators — 136
2.3.1	Symbols and characterization of Laplacian modulated operators — 136
2.3.2	Examples of Laplacian modulated operators — 141
2.3.3	Pseudodifferential operators are Laplacian modulated — 143
2.4	Eigenvalues of Laplacian modulated operators — 145
2.5	Trace formulas for Laplacian modulated operators — 152
2.5.1	Computing traces from symbols — 152
2.5.2	Connes' trace theorem — 159
2.6	Notes — 162
3	Integration formulas — 167
3.1	Introduction — 167
3.2	Integration formula for elliptic differential operators and the curved plane — 170
3.2.1	Integration formula for elliptic differential operators — 171
3.2.2	Integration theorem on a curved plane — 177
3.3	Integration of square-integrable functions on $\mathbb{T}^d$ — 180
3.4	Integration of compactly supported square-integrable functions on $\mathbb{R}^d$ — 185
3.5	Integration formula on the noncommutative torus — 190
3.5.1	Definition of the noncommutative torus — 190
3.5.2	Differential calculus of the noncommutative torus — 191
3.5.3	Product-convolution operators for the noncommutative torus — 192
3.5.4	Integration of square-integrable functions on the noncommutative torus — 196
3.6	Notes — 197

Part II: The principal symbol mapping in noncommutative geometry

4	Integration formula for the noncommutative plane — 209
4.1	Introduction — 209
4.2	Definition of the noncommutative plane — 214
4.2.1	Canonical commutation relations — 214
4.2.2	Distributions and the Schwartz space of the noncommutative plane — 222
4.2.3	Differentiation for the noncommutative plane — 231
4.2.4	Additional symmetries of the noncommutative plane — 239
4.3	Product-convolution estimates for the noncommutative plane — 241
4.4	Smooth product-convolution estimates for the noncommutative plane — 251
4.4.1	Integral formula for derivations of operators acting on invariant subspaces — 252
4.4.2	Product-convolution estimates for partial derivations on the noncommutative plane — 257
4.5	Integration formula up to a constant — 260
4.6	Measurability in the noncommutative plane — 266
4.7	Notes — 276
5	A C^*-algebraic approach to principal symbols and trace formulas — 278
5.1	Introduction — 278
5.2	Principal symbol mapping in the C^* -algebraic context — 281
5.2.1	C^* -norms on tensor products of C^* -algebras — 282
5.2.2	Principal symbol mapping using the Calkin quotient map — 283
5.2.3	An abstract trace formula — 286
5.3	Trace formula for zero-order pseudodifferential operators — 289
5.3.1	Symbol map on product-convolution operators on $\mathbb{R}^d$ — 290
5.3.2	Extension of Connes' trace theorem — 297
5.3.3	Failure of a symbol map for nonhomogeneous operators — 299
5.4	Principal symbol and trace formula for noncommutative tori — 302
5.4.1	Principal symbol map — 302
5.4.2	Trace formula — 310
5.5	Principal symbol and trace formula for the noncommutative plane — 314
5.5.1	Principal symbol map — 314
5.5.2	Trace formula — 321
5.6	Principal symbol and trace formula for $SU(2)$ — 330
5.6.1	Differential calculus of $SU(2)$ — 330
5.6.2	Principal symbol map — 334
5.6.3	Description of the symbol algebra for $SU(2)$ — 338
5.6.4	Trace formula — 344

5.7 Notes — **347**

6 Quantum differentiability for the Euclidean plane — 349

6.1 Introduction — **349**

6.2 Compactness estimates for the quantized differential of a differentiable function — **354**

6.3 Approximation of quantized differentials by principal terms — **361**

6.3.1 Approximation of differentials — **361**

6.3.2 Approximation of densities — **367**

6.4 Integration of quantum densities — **371**

6.5 Approximation of homogeneous Sobolev functions by smooth functions — **377**

6.6 Trace formula for quantum densities on the Euclidean plane — **381**

6.7 Notes — **386**

Part III: Further applications

7 Connes character formula — 391

7.1 Introduction — **391**

7.1.1 Spectral triples and the Connes–Chern character — **391**

7.1.2 Hochschild homology and cohomology — **394**

7.1.3 Statement of the Connes character formula — **395**

7.2 Exploiting Hochschild cohomology — **397**

7.3 Commutator estimates — **406**

7.4 Asymptotics for the heat semigroup — **412**

7.5 Proof of the Connes character formula — **418**

7.6 Notes — **425**

8 Density of states — 426

8.1 Introduction — **426**

8.2 Compactness estimate for the density of states — **433**

8.3 The density of states trace formula — **442**

8.4 Invariance under perturbation — **448**

8.5 Density of states for homogeneous potentials — **451**

8.6 Notes — **454**

Bibliography — 457

Index — 469