

Contents

Part I Introduction

1	Introductory Remarks	3
1.1	From Spherical Harmonics to Zonal Kernels	5
1.2	From Scalar Spherical Harmonics to its Vectorial and Tensorial Extensions	16
1.3	Organization of the Work	23

Part II Background and Nomenclature

2	Basic Settings	31
2.1	Euclidean Nomenclature	31
2.2	Euclidean Differential Calculus	37
2.3	Euclidean Integral Calculus	40
2.4	Spherical Nomenclature	43
2.5	Spherical Differential Calculus	47
2.6	Spherical Integral Calculus	52
2.7	Orthogonal Invariance	62
3	Circular Harmonics	71
3.1	Homogeneous Harmonic Polynomials and Circular Harmonics	72
3.2	Eigenfunctions of the Beltrami Operator	72
3.3	Addition Theorem and Chebyshev Polynomial	73
3.4	Stereographic Projection	75
3.5	Closure and Completeness of Circular Harmonics	77
3.6	Inner/Outer Harmonics	78

Part III Spherical Harmonics

4	Scalar Spherical Harmonics	83
4.1	Homogeneous Harmonic Polynomials	84
4.2	Addition Theorem	92

4.3	Exact Computation of Homogeneous Harmonic Polynomials	98
4.4	Definition of Scalar Spherical Harmonics	108
4.5	Legendre Polynomials	112
4.6	Orthogonal (Fourier) Expansions	122
4.7	Legendre (Spherical) Harmonics	136
4.8	Funk-Hecke Formula	140
4.9	Eigenfunctions of the Beltrami Operator	143
4.10	Irreducibility of Scalar Harmonics	144
4.11	Degree and Order Variances	148
4.12	Associated Legendre Polynomials	154
4.13	Associated Legendre (Spherical) Harmonics	163
4.14	Spherical Harmonics in Cartesian Coordinates	171
4.15	Spherical Harmonics in Terms of Wigner D -Matrices	184
4.16	Bibliographical Notes	191
5	Green's Functions and Integral Formulas	193
5.1	Green's Function with Respect to the Beltrami Operator	193
5.2	Space Mollifier Green's Function with Respect to the Beltrami Operator	196
5.3	Frequency Mollifier Green's Function with Respect to the Beltrami Operator	204
5.4	Modified Green Functions	208
5.5	Integral Formulas	212
5.6	Differential Equations	216
5.7	Classical Understanding of Spline Functions	218
5.8	Integral Formulas with Respect to Iterated Beltrami Operators	224
5.9	Differential Equations Respect to Iterated Beltrami Operators	233
5.10	Pseudodifferential Operators, Green's Functions, and Splines	236
5.11	Bibliographical Notes	249
6	Vector Spherical Harmonics	251
6.1	Normal and Tangential Fields	253
6.2	Helmholtz System of Vector Spherical Harmonics	254
6.3	Orthogonal (Fourier) Expansions	263
6.4	Homogeneous Harmonic Vector Polynomials	271
6.5	Exact Computation of Orthonormal Systems	274
6.6	Irreducibility and Orthogonal Invariance of Vector Spherical Harmonics	280
6.7	Vectorial Beltrami Operator	288
6.8	Vectorial Addition Theorem	290
6.9	Vectorial Funk-Hecke Formulas	297
6.10	Counterparts to the Legendre Polynomial	301
6.11	Degree and Order Variances	305
6.12	Hardy-Hodge System of Vector Spherical Harmonics	309
6.13	Orthogonal Expansion Using Vector Legendre Kernels	318

6.14	Bibliographical Notes	323
7	Tensor Spherical Harmonics	325
7.1	Some Nomenclature	326
7.2	Normal and Tangential Fields	328
7.3	Integral Theorems	331
7.4	Helmholtz System of Tensor Spherical Harmonics	335
7.5	Helmholtz Decomposition Theorem	342
7.6	Closure and Completeness of Tensor Spherical Harmonics	346
7.7	Homogeneous Harmonic Tensor Polynomials	354
7.8	Tensorial Beltrami Operator	360
7.9	Tensorial Addition Theorem	363
7.10	Tensorial Funk-Hecke Formulas	372
7.11	Counterparts to the Legendre Polynomials	377
7.12	Tensor Spherical Harmonics Related to Tensor Homogeneous Harmonic Polynomials	379
7.13	Hardy-Hodge System of Tensor Spherical Harmonics	383
7.14	Orthogonal Expansion Using Tensor Legendre Kernels	389
7.15	Bibliographical Notes	392
8	Spin-Weighted Spherical Harmonics	395
8.1	Spin-Weighted Differential Calculus	396
8.2	Spin-Weighted Integral Calculus	399
8.3	Spin-Weighted Spherical Harmonics in Coordinate Representation ..	409
8.4	Spin-Weighted Spherical Harmonics and the Wigner D -Function ..	423
8.5	Relations to Vector and Tensor Spherical Harmonics	453
8.6	Bibliographical Notes	456

Part IV Zonal Functions

9	Scalar Zonal Kernel Functions	461
9.1	Bandlimited/Spacelimited Functions	462
9.2	Zonal Kernel Functions	462
9.3	Zonal Kernel Functions in Scalar Context	466
9.4	Convolutions Involving Scalar Zonal Kernel Functions	467
9.5	Classification of Zonal Kernel Functions	469
9.6	Localization of Representative Zonal Kernels	478
9.7	Dirac Families of Zonal Scalar Kernel Functions	491
9.8	Examples of Dirac Families	500
9.9	Bibliographical Notes	520

10	Vector Zonal Kernel Functions	523
10.1	Preparatory Material	524
10.2	Tensor Zonal Kernel Functions of Rank Two in Vectorial Context	525
10.3	Vector Zonal Kernel Functions in Vectorial Context	531
10.4	Convolutions Involving Vector Zonal Kernel Functions	533
10.5	Dirac Families of Zonal Vector Kernel Functions	535
10.6	Bibliographical Notes	537
11	Tensorial Zonal Kernel Functions	539
11.1	Preparatory Material	540
11.2	Tensor Zonal Kernel Functions of Rank Four in Tensorial Context	540
11.3	Convolutions Involving Zonal Tensor Kernel Functions	542
11.4	Tensor Zonal Kernel Functions of Rank Two in Tensorial Context	544
11.5	Dirac Families of Zonal Tensor Kernel Functions	548
11.6	Bibliographical Notes	549
 Part V Geopotentials		
12	Potentials in Euclidean Space	553
12.1	Newton's Law of Gravitation	553
12.2	Volume Potential	554
12.3	Interior/Exterior Green's Formula	558
12.4	Volume Potential and Gravitational Field	565
12.5	Scalar Inner/Outer Harmonics	570
12.6	Surface Potentials	575
12.7	Boundary-Value Problems for the Laplace Operator	578
12.8	Existence and Regularity Method	590
12.9	Locally and Globally Uniform Approximation	595
12.10	Vector Outer Harmonics and the Gravitational Gradient	612
12.11	Satellite-to-Satellite Tracking	616
12.12	Tensor Outer Harmonics and the Gravitational Tensor	620
12.13	Satellite Gravitation Gradiometry	623
12.14	Function Systems for Internal Elastic Fields	628
12.15	Bibliographical Notes	648
13	Potentials on the Sphere	651
13.1	Green's Formulas	651
13.2	Surface Harmonicity	653
13.3	Surface Potentials	654
13.4	Curve Potentials	656
13.5	Limit Formulas and Jump Relations	656
13.6	Boundary-Value Problems for the Beltrami Operator	657
13.7	Complete Function Systems	663
13.8	Bibliographical Notes	667

14 Multiscale Mollifier Potential Systems in Geoapplications 669

14.1 Surface Mollifier Gradient Fields and Gravitational Deflections of
the Vertical 669

14.2 Surface Mollifier Curl Gradient Fields and Geostrophic Ocean Flow 680

14.3 Bibliographical Notes 686

Part VI Conclusion

15 Concluding Remarks 689

List of Symbols 697

References 703

Index 723