## Karl-Georg Steffens

## The History of Approximation Theory

From Euler to Bernstein

Birkhäuser Boston • Basel • Berlin

## Contents

Pr	eface			V
Int	rodu	ction .		vii
1	For	erunne	ers	1
	1.1	Euler'	s Analysis of Delisle's Map	1
		1.1.1	The Delislian Projection	1
		1.1.2	Euler's Method	3
		1.1.3	Determining the Intersecting Parallels $P$ and $Q$	5
		1.1.4	Minimization of the Error of the Projection	6
		1.1.5	Discussion	8
	1.2	Laplac	ce's Approximation of Earth's Surface	9
		1.2.1	A Formula to Compute a Part of the Arc of an Ellipse .	9
			1.2.1.1 The arc length of a part of an ellipse	10
		1.2.2	A Characteristic System of Equations	11
		1.2.3	A First Solving Algorithm	12
		1.2.4	A Second Algorithm and a Necessary Alternation	
			Condition	13
		1.2.5	Determination of the Maximal Errors	15
			1.2.5.1 Determination of the Largest Error	15
			1.2.5.2 Determination of the Smallest Error	17
			1.2.5.3 Determination of the Best Ellipse	17
		1.2.6	Application to Geodesy	18
		1.2.7	A Discrete Approximation Problem	19
2	Paf	nuti L	vovich Chebyshev	21
	2.1		shev's Curriculum Vitae	
	2.2	Stimu	ı v	
		2.2.1	Chebyshev's Trip Abroad	
		2.2.2	Poncelet's Approximation Formulae	
		2.2.3	Watt's Mechanism	27

3

	2.2.4	Watt's Curve	27
2.3	First 7	Theoretical Approaches	30
	2.3.1	Characteristic Equations	
	2.3.2	Approaches for Real-Analytic Functions	33
	2.3.3	The Polynomial of $(n + 1)$ thDegree and Given First	
		Coefficient Least Deviating from Zero	34
		2.3.3.1 Determination of $T_n$ . The Case $m = 0 \dots$	35
	2.3.4	Remarks on "Théorie des mécanismes"	36
2.4	First 7	Theoretical Compositions	37
	2.4.1	Questions about Minima	37
	2.4.2	A General Necessary Criterion	38
	2.4.3	The Number of Deviation Points. Three Cases	40
	2.1.0	2.4.3.1 Approximation by Polynomials	41
	2.4.4	Determination of $T_n$	42
		2.4.4.1 Expansion of continous fractions	43
	2.4.5	Solvability of Algebraic Equations	45
	2.4.6	Application to Interpolation Problems	47
	2.4.7	Evaluation of 'Questions about Minima'	48
	2.4.8	Chebyshev's Aim	50
2.5		y of Orthogonal Polynomials	51
	2.5.1	On Continuous Fractions	51
	2.5.2	Chebyshev-Fourier Series	56
	2.5.3	Theory of Jacobian Polynomials	57
	2.5.4	Approximation Preserving Monotonicity	58
		2.5.4.1 Preconsiderations	58
		2.5.4.2 Calculation of the Minimal Solution	59
		2.5.4.3 Calculation of the Minimal Deviation	64
2.6	Other	Contributions of P. L. Chebyshev	67
	2.6.1	Theory of Mechanisms	67
	2.6.2	Geodesy and Cartography	68
	2.6.3	Approximated Quadrature Formulae	68
2.7	Cheby	shev–Euler of the 18th Century?	69
		Petersburg Mathematical School	77
3.1		andr Nikolaevich Korkin	79
	3.1.1	About Korkin's Biography	
	3.1.2	The Scientific Work of A. N. Korkin	81
	3.1.3		83
3.2	Egor I	vanovich Zolotarev	84
	3.2.1	Biographical Data	84
	3.2.2	Application of the Theory of Elliptic Functions to	
		Approximation Theory	85
		3.2.2.1 Zolotarev's solutions	87
	3.2.3	$L_1$ -Approximation	90
		3 2 3 1 An excursion into Laurent coefficients	92

	Contents	xvii
	3.2.4 Zolotarev's Conceptional Apparatus	. 96
3.3	Andrey and Vladimir Andreevich Markov	
	3.3.1 About Andrey Andreevich Markovs Life	
	3.3.2 The Early Work on Approximation Theory	
	3.3.2.1 About a Question by D. I. Mendeleev. An	
	Alternation Theorem	100
	3.3.2.1.1 First Problem	
	3.3.2.2 Markov Systems	
	3.3.2.3 The Lecturer Andrey Markov	
	3.3.3 Vladimir Andreevich Markov	
	3.3.3.1 About his Biography	
	3.3.3.2 A Student's Paper	
	3.3.3.3 Vladimir Markov's Problem	
	3.3.3.4 An Alternation Theorem by V. A. Markov	
	3.3.3.5 The Special Side-Condition $\omega(p) = p^{(k)} \ldots$	
3.4	Julian Karol Sochocki	
	3.4.1 About his Biography	. 111
	3.4.2 Chebyshev's Supplement	. 113
3.5	Konstantin Aleksandrovich Posse	. 113
	3.5.1 His Biography	. 114
	3.5.2 The Scientific Work of K. A. Posse	. 116
	3.5.2.1 Orthogonal Polynomials	. 116
	3.5.2.2 Other Works	. 117
	3.5.2.3 Posse's Basic Concepts	. 118
3.6	A. A. Markov's Lectures	. 119
	3.6.1 Circumstances and Significance	
	3.6.2 The Posing of the General Problem	
	3.6.3 General Results	
	3.6.4 Polynomial Approximation	. 122
	3.6.5 Extremal Problems: Polynomials Deviating the Least	
	Possible from Zero	
	3.6.6 An Interpolatory Side-Condition	
	3.6.7 The Main Subject of A. Markov's Lectures	
3.7	Résumé: Practice by Algebraic Methods	
	3.7.1 Decadency and Pseudo-Geometry	128
Dev	velopment Outside Russia	. 131
4.1	The Mediator: Felix Klein	
4.2	Blichfeldt's Note	. 135
4.3	Kirchberger's Thesis	
	4.3.1 Existence, Uniqueness and Continuity	
	4.3.1.1 Return to Weierstraß' Fundamental Theorem.	
	4.3.1.2 Proof of the Alternation Theorem	. 138
	4.3.2 Continuity of the Operator of Best Approximation	
	4.3.3 Rational Approximation	. 141

		4.3.4	A Discrete Approximation Problem	142
		4.3.5	An Algorithm for the Approximate Determination of	
			the Best Approximation of Real-Analytic Functions	142
		4.3.6	Approximation under Side-Conditions	
		4.3.7	Foundations for Chebyshev's Methods	
	4.4	Other	Non-Quantitative Contributions	
		4.4.1	Borel	145
		4.4.2	Young's Systems	146
		4.4.3	Trigonometric Approximation	147
	4.5	On Co	onvergence and Series Expansions	147
		4.5.1	Weierstraß' Approximation Theorem	
	4.6	Fejér	and Runge	
		4.6.1	Summable Fourier Series	
		4.6.2	Runge's Ideas about Approximation Theory	152
			4.6.2.1 Approximation by Representation	152
			4.6.2.2 Expansion after Chebyshev Polynomials	
			4.6.2.3 A Special Concept for Applied Mathematics—	
			'Sensible Functions'	154
	4.7	Quant	titative Approximation Theory	158
	4.8		on's Thesis	
		4.8.1	Jackson's Theorem	162
		4.8.2	Further Results. An Inverse Theorem	163
		4.8.3	How the Faculty Judged	164
	4.9	A Not	te About Göttingen's Rôle	165
_	~		ar in ar one ize ar	1.05
5			tive Function Theory: Kharkiv	
	5.1		ni-Bonifatsi Pavlovich Psheborski	
		5.1.1	His Biography	
		5.1.2	Psheborski in Göttingen	
		5.1.3	Doctoral Thesis and Lecturer's Job	
		5.1.4	An Extension of V. A. Markov's Problem	
			5.1.4.1 Existence of a Solution	
	T 0	A C1-	5.1.4.2 Criteria for the Solution	
	5.2		ort Biography of Sergey Natanovich Bernstein	
	5.3		Contributions to Approximation Theory	
		5.3.1	A Proof of Weierstrass' Theorem	178
		5.3.2	A Prize Competition of the Belgian Academy of Sciences	100
		F 9 9		
		5.3.3 5.3.4	The Prize-Winning Treatise	186
		ა.ა.4	Jackson's and Bernstein's Contributions	104
		5.3.5	Quasianalytic Functions	
	5.4		•	187
	5.4		ructive Function Theory as the Development of vshev's Ideas	100
		Cheby	vonev o ideas	105

		Contents	xix
A	Biographies		191
	A.1 Matvey Aleksandrovich Tikhomandritski		
	A.2 Nikolay Yakovlevich Sonin		
	A.3 Aleksandr Vasilevich Vasilev		
	A.4 Ivan Lvovich Ptashitski		193
	A.5 Dmitri Fedorovich Selivanov		194
	A.6 Aleksandr Mikhaylovich Lyapunov		195
	A.7 Ivan Ivanovich Ivanov		196
	A.8 Dmitri Aleksandrovich Grave		196
	A.9 Georgi Feodosievich Voronoy		197
В	Explanations		199
	B.1 Russian Academic Degrees		
Bib	oliography		201
T d	los-		217