

Joe Iannelli

Characteristics Finite Element Methods in Computational Fluid Dynamics

With 384 Figures

 Springer

Contents

Preface

VII

A Brief History of Theoretical and Computational Fluid Dynamics

XXIII

1	Governing Equations of Fluid Mechanics	1
1.1	Fluid Particle	1
1.2	Mathematical Description of Fluid Flow	4
1.3	Transport Theorem	8
1.3.1	Surface Integral Form	9
1.3.2	Divergence Form	11
1.4	Flowing and Stationary Fluid Volumes	13
1.5	Stress Tensor	13
1.5.1	Geometry of a Tetrahedron	13
1.5.2	Surface Traction	15
1.5.3	Stress Tensor Symmetry	16
1.5.4	Normal Stresses	17
1.5.5	Stress Invariant, Mechanical Pressure and Deviatoric Stresses	18
1.6	Flow State Variables	19
1.7	Fundamental Equations	20
1.7.1	Conservation of Mass: The Continuity Equation	20
1.7.2	Second Law of Mechanics: The Linear Momentum Equation	21
1.7.3	The First Law of Thermodynamics: The Total Energy Equation	23
1.8	Consequences of Mass Conservation	26
1.8.1	Substantive Derivative	26
1.8.2	Substantive-Derivative Form of The Transport Theorem	27
1.8.3	Interchange of Differentiation Order for Mass and Time	27
1.9	Velocity Divergence	28
1.9.1	Physical Significance	28
1.9.2	Equations for a Fluid Particle	28
1.10	Energy Equations	29
1.10.1	Kinetic Energy Equation	30
1.10.2	Rigid-Motion and Deformation Work	31
1.10.3	Internal Energy Equation	33
1.10.4	Entropy Equation	33
1.10.5	Total Enthalpy Equation	35
1.10.6	Internal Enthalpy and Temperature Equations	35
1.11	Special Solutions of the Internal-Energy, Enthalpy, and Momentum Equations	36

1.11.1	Isentropic Solution of the Internal Energy Equation	36
1.11.2	Adiabatic- and Inviscid-Flow Solution of the Total-Enthalpy Equation	37
1.11.3	Isentropic Solution of the Momentum Equation	38
2	Constitutive and State Equations	41
2.1	Navier-Stokes Stress Tensor	41
2.2	Heat Conduction Law	43
2.3	Transport and Thermo-Mechanical Properties	43
2.4	Second-Law Constraints on Transport Coefficients	45
2.5	Equations of State	46
2.6	Speed of Sound	47
2.7	Isentropic Constraint on Derivatives of Density and Energy	48
3	State Equations for Reacting Air	51
3.1	Thermo-Chemical Equations	52
3.1.1	Species Masses and Moles	52
3.1.2	Mass and Mole Fractions	53
3.1.3	Pressure, Internal-Energy, Mass-Fraction Equations	54
3.1.4	Perfect-Air Equations	56
3.1.5	Non-dimensional Reacting-Air Equations of State	58
3.1.6	Chemical and Thermodynamics Parameters	61
3.1.7	Chemical Equilibrium Functions	62
3.2	Algebraic Solution	63
3.3	Calculation of Y_3 and T	65
3.3.1	Numerical Solution	65
3.3.2	Iteration Partial Derivatives	66
3.4	Partial Derivatives of $T, Y_1, Y_2, Y_3, Y_4, Y_5$ with Respect to ρ and ϵ	69
3.4.1	Derivatives of T and Y_3	69
3.4.2	Derivatives of Y_1, Y_2, Y_4, Y_5	70
3.5	Pressure and its Thermodynamic Derivatives	72
3.6	Jacobian Partial Derivatives of Pressure and Temperature	72
3.7	Derivatives of p and Y_i with Respect to ρ, T , Derivatives of ρ with Respect to p and T , and Derivatives of T with Respect to ρ and p	73
3.8	Thermodynamic Properties with p and T as Independent Variables	74
3.9	Specific Heats	75
3.9.1	Specific Heat c_v	76
3.9.2	Specific Heat c_p	76
3.10	Speed of Sound	78
3.11	Computational Results	79
3.12	Performance Summary	91
4	Euler and Navier Stokes Systems	93
4.1	Compressible Flows	95
4.1.1	Conservation Form	95
4.1.2	Non-Dimensional System	96

4.1.3	Variational Formulation	98
4.2	Low Speed and Incompressible Flows	98
4.2.1	Continuity Equation	99
4.2.2	Momentum Equation	99
4.2.3	Temperature Equation	100
4.2.4	Governing System	103
4.2.5	Non-Dimensional System	103
4.2.6	Variational Formulation	105
5	Quasi One-Dimensional and Free-Surface Equations	107
5.1	Area Averages	107
5.2	Quasi One-Dimensional Euler Equations	108
5.2.1	Continuity Equation	108
5.2.2	Generalized Linear-Momentum Equation	110
5.2.3	Energy Equation	114
5.2.4	Compressible-Flow System	118
5.2.5	Incompressible-Flow System	120
5.3	Open-Channel Configuration	121
5.3.1	Depth Averages	122
5.3.2	Area Averages	123
5.4	Quasi One-Dimensional Free-Surface Open-Channel Flow Equations	125
5.4.1	Continuity Equation	126
5.4.2	Momentum Equation	128
5.4.3	Energy Equation	132
5.4.4	Quasi One-Dimensional Free-Surface System	135
5.5	Two-Dimensional Free-Surface Equations	137
5.5.1	Continuity Equation	137
5.5.2	Momentum Equation	137
5.5.3	Energy Equation	142
5.5.4	Non-Dimensional 2-D Free-Surface Equations	146
6	Overview of CFD Algorithm Development	149
6.1	Mathematical Model	150
6.2	Spatial Approximation: Discretization	154
6.2.1	Steady-Solution Monotonicity	155
6.2.2	Unsteady-Solution Stability	157
6.3	Time Integration	160
6.4	Solution of Non-Linear Systems	164
7	The Finite Element Method	167
7.1	Variational Weighted Integral Statements	169
7.1.1	Least Squares Statements	169
7.1.2	Representative Optimal Solutions	171
7.1.3	Fluid Dynamics Equations from a Variational Principle	173
7.2	Integrals from Gauss' Divergence Theorem	175

7.3	Weak Derivatives and Integral Statement	177
7.4	Characteristics-Bias Formulation	178
7.4.1	Non-Discrete Discontinuous Galerkin (DG) Form	179
7.4.2	Non-Discrete Streamline Upwind Petrov-Galerkin (SUPG) Form	180
7.5	Hilbert Spaces and Approximation Results	181
7.5.1	Hilbert Spaces	181
7.5.2	Approximation Results	182
7.6	Curvilinear Coordinate Transformations	182
7.6.1	Areas and Volumes	182
7.6.2	Two-Dimensional Coordinate Transformation	184
7.6.3	Three-Dimensional Coordinate Transformation	187
7.6.4	Differential Arc, Area, Volume	191
7.7	Discretized Statement	194
7.7.1	Differential and Integral Curvilinear-Coordinate Statements	194
7.7.2	Finite Element Galerkin Formulation	196
7.7.3	Discretization of Space	199
7.7.4	Discretization of Functions	200
7.7.5	A-Posteriori Accuracy Assessment	203
7.8	Basis Functions, Element and Coordinate Transformations	205
7.8.1	One-Dimensional Elements and Coordinate Transformations	205
7.8.2	Multi-Dimensional Elements	208
7.9	Element Integrals, Gaussian Quadratures, Optimal Metric Data	211
7.9.1	One-Dimensional Integrals	211
7.9.2	Gaussian Quadratures	212
7.9.3	Multi-Dimensional Integrals and Optimal Metric Data	213
7.10	Implementation Sequence	217
8	Non-Linearly Stable Implicit Runge-Kutta Time Integrations	225
8.1	Runge-Kutta Algorithm	226
8.2	Non-Linear Stability	227
8.3	Linear Stability	230
8.4	IRK2: A Two-Stage Diagonally Implicit Algorithm	231
8.4.1	Numerical Linear Algebra	231
8.4.2	Stability Constraints	232
8.4.3	Accuracy and B - Consistency / Convergence Constraints	233
8.4.4	Solution of the IRK Coefficient Equations	234
8.4.5	Accuracy and Stability Performance	235
8.5	One- and Two-Stage Diagonally Implicit Algorithms	237
8.6	Time Step Size Adjustment	237
8.7	Theoretical Comparisons	239
9	One-Dimensional Non-Discrete Characteristics-Bias Resolution	241
9.1	Solutions of Non-Linear Hyperbolic Equations	242
9.1.1	Solutions and Characteristic Lines	243
9.1.2	Breaking Time	246

9.1.3	Shocks	247
9.1.4	Hyperbolic Systems	249
9.2	Parabolic Perturbation Equations	250
9.2.1	Scalar Equations	250
9.2.2	Perturbation Systems	252
9.3	Characteristics-Bias Resolution	254
9.3.1	Upstream-Bias Time Derivative and Source	254
9.3.2	Flux Jacobian Decomposition and Upstream-Bias Integral Average	255
9.3.3	Characteristics-Bias Flux	255
9.3.4	Flux Vector Splitting	256
9.3.5	Flux Difference Splitting	257
9.3.6	Characteristics-Bias System	258
9.4	Galilean Invariance and Characteristics-Bias Diffusion	259
9.5	Non-Discrete Discontinuous Galerkin (DG) Form	262
9.6	Non-Discrete Streamline Upwind Petrov-Galerkin (SUPG) Form	263
10	Characteristics-Bias Controller and Length	265
10.1	Measure of Solution Smoothness	265
10.1.1	One-Dimensional Flows	267
10.1.2	Multi-Dimensional Flows	268
10.2	Correlation between φ and ψ	270
10.3	Multi-Dimensional Length ε	273
10.4	Magnitude of φ , ψ , $\varepsilon\psi$	276
11	Computational Analysis of Quasi-1-D Incompressible Flows	279
11.1	Slight-Compressibility Continuity Equation	280
11.2	Incompressible Flow "Characteristics"	281
11.3	Slight-Compressibility Acoustics Equations	283
11.4	Incompressible-Flow Acoustics-Convection System-Matrix Decomposition	284
11.4.1	Convection and Pressure-Jacobian Components	285
11.4.2	Acoustic Components	285
11.4.3	Combination of Matrix Decompositions	286
11.5	Characteristics-Bias System Matrix	287
11.5.1	Consistent Upstream-Bias	288
11.5.2	Upstream-Bias Eigenvalues and Functions α and δ	288
11.6	Incompressible-Flow Formulation	290
11.7	Variational Statement and Boundary Conditions	292
11.8	Finite Element Weak Statement	294
11.8.1	Galerkin Finite Element Expansions and Integrals	295
11.8.2	Discrete Upstream Bias	297
11.8.3	Boundary Equations and Pressure Boundary Condition	298
11.9	Implicit Runge Kutta Time Integration	299
11.10	Reference Exact Solutions	302
11.10.1	General Solution	302
11.10.2	Particular Solution	303

11.10.3	Solution Properties	303
11.10.4	Transient Solution	304
11.10.5	Steady State Solution	305
11.10.6	Similarity Solution	307
11.11	Computational Results	308
11.11.1	Unsteady Flow: Periodic Outlet Pressure	310
11.11.2	Transient and Steady State Flow	314
11.12	Computational Performance	319
12	Numerical Study of Generalized Quasi-1-D Free Surface Flows	321
12.1	One-Dimensional Free-Surface Flow Characteristics	322
12.2	Gravity-Wave Equations	325
12.3	Free-Surface Gravity-Wave-Convection Flux Jacobian Decomposition	327
12.3.1	Convection and Pressure-Gradient Components	327
12.3.2	Gravity-Wave Components	328
12.3.3	Combination of Flux-Jacobian Decompositions	330
12.4	Characteristics-Bias Flux Divergence	331
12.4.1	Consistent Upstream-Bias	331
12.4.2	Upstream-Bias Eigenvalues and Functions α and δ	332
12.5	Variational Statement and Boundary Conditions	335
12.6	Finite Element Weak Statement	338
12.6.1	Galerkin Finite Element Expansions and Integrals	339
12.6.2	Discrete Upstream Bias	340
12.6.3	Boundary Equations and Pressure Boundary Conditions	342
12.7	Implicit Runge Kutta Time Integration	343
12.8	Reference Exact Solutions	345
12.8.1	Channel-Width Rule	345
12.8.2	Exact Integrals	346
12.8.3	Similarity Solution	347
12.8.4	Hydraulic Jump Calculation	348
12.9	Computational Results	351
12.9.1	Work Driven Flows	352
12.9.2	Mass-Transfer Flows	354
12.9.3	Adiabatic Smooth and Shocked Flows with Friction	357
12.9.4	Shocked Flows with Friction, Mass Transfer and Bed Slope	361
12.9.5	Smooth and Shocked Flows over a Bump	364
12.9.6	Dam-Break Flow	370
12.9.7	Flow with Hydraulic Jumps in a Variable-Width Channel	376
12.9.8	Flow with Hydraulic Jumps in a Variable-Width Channel with Friction, Sloping Bed, Mass as well as Heat Transfer	382
12.9.9	Convergence to Steady State	385
12.10	Computational Performance	386

13 CFD Investigation of Generalized Quasi-1-D Compressible Flows	389
13.1 Characteristics Analysis	390
13.2 Acoustics Equations	393
13.3 Acoustics-Convection Flux Jacobian Decomposition	395
13.3.1 Convection and Pressure-Gradient Components	395
13.3.2 Acoustic Components	396
13.3.3 Combination of Flux-Jacobian Decompositions	398
13.4 Characteristics-Bias Flux-Divergence	399
13.4.1 Consistent Upstream-Bias	399
13.4.2 Upstream-Bias Eigenvalues and Functions α and δ	400
13.5 Variational Statement and Boundary Conditions	403
13.6 Finite Element Weak Statement	406
13.6.1 Galerkin Finite Element Expansions and Integrals	407
13.6.2 Discrete Upstream Bias	408
13.6.3 Boundary Equations and Pressure Boundary Condition	410
13.7 Implicit Runge Kutta Time Integration	411
13.8 Reference Exact Solutions	413
13.8.1 Smooth-Flow Pressure Integral	414
13.8.2 Steady-State Equations	415
13.8.3 Area Rule	415
13.8.4 Exact Integrals	416
13.8.5 Similarity Solution	417
13.8.6 Calculation of Shocked Flows	420
13.9 Computational Results	423
13.9.1 Sod's Shock Tube Flow	424
13.9.2 Heat-Transfer Flows	427
13.9.3 Mass-Transfer Flows	429
13.9.4 Mass- and Heat-Transfer Flows	431
13.9.5 Work Driven Flows	433
13.9.6 Adiabatic Smooth and Shocked Flows with Wall Friction	435
13.9.7 Shocked Flows with Wall Friction, Heat, Mass, and Work Transfer	439
13.9.8 Shapiro's Benchmark	441
13.9.9 Flows in a Diverging Nozzle	443
13.9.10 Adiabatic Smooth and Shocked Flows in a Converging-Diverging Nozzle	448
13.9.11 Shocked Nozzle Flows with Wall Friction, Heating, Mass and Work Transfer	454
13.9.12 Convergence to Steady State	458
13.10 Computational Performance	459
14 Multi-Dimensional Characteristics and Characteristics-Bias Systems	461
14.1 Hyperbolic Equations and Elliptic, Parabolic, Hyperbolic Systems	462
14.1.1 Scalar Equations	462
14.1.2 Non-linear Wave-Like Vector Solutions	464
14.1.3 Non-Linear Elliptic, Parabolic, Hyperbolic Systems	467
14.2 Wave-Propagation and Characteristic Surfaces	468

14.2.1	Characteristic Surfaces	468
14.2.2	Characteristic and Discontinuity Surfaces	469
14.3	Characteristic Cones and Hyperbolic Wave Propagation	470
14.3.1	Galilean Transformation	470
14.3.2	Characteristic Cones and Wave Propagation Patterns	472
14.4	Parabolic Perturbation System	477
14.5	Characteristics-Bias Representation	480
14.5.1	Upstream-Bias Time Derivative, Source, Viscous-Flux Divergence	481
14.5.2	Flux Jacobian Decomposition and Upstream-Bias Integral Average	482
14.5.3	Characteristics-Bias Flux	483
14.5.4	Upstream-Bias Stability Condition	483
14.5.5	Incorporation of Flux-Vector and Flux-Difference Splittings	485
14.5.6	Characteristics-Bias System	486
14.6	Galilean Invariance and Characteristics-Bias Diffusion	487
14.7	Non-Discrete Discontinuous Galerkin (DG) Form	491
14.8	Non-Discrete Streamline Upwind Petrov-Galerkin (SUPG) Form	491
15	Multi-Dimensional Incompressible Flows	493
15.1	Slight-Compressibility System	494
15.1.1	Slight-Compressibility Continuity Equation	494
15.1.2	Euler and Navier-Stokes Equations	495
15.2	Slight-Compressibility Acoustics Equations	496
15.2.1	Matrix Form	496
15.2.2	Streamline and Crossflow Components	498
15.2.3	Similarity Transformation	501
15.3	Characteristic Analysis and Velocity Components	503
15.3.1	Polar Variation of Characteristic Speeds	504
15.3.2	Wave Propagation Region and Characteristic Cone	506
15.3.3	Physical Multi-dimensional Upstream Bias	510
15.4	Acoustics-Convection Characteristics-Bias Decomposition	511
15.4.1	2-D Convection and Pressure-Gradient Components	511
15.4.2	2-D Acoustic Components	512
15.4.3	Combination of 2-D Acoustics-Convection Decompositions	514
15.4.4	3-D Convection and Pressure-Gradient Components	515
15.4.5	3-D Acoustic Components	517
15.4.6	Combination of 3-D Acoustics-Convection Decompositions	519
15.5	Characteristics-Bias System Matrix	520
15.5.1	Consistent Multi-Dimensional and Infinite-Directional Upstream Bias	522
15.5.2	Upstream-Bias Eigenvalues	523
15.5.3	Conditions on Upstream-Bias Functions and Eigenvalues	523
15.5.4	Upstream-Bias Vectors $\mathbf{a}$, $\mathbf{a}^{N_1}$, $\mathbf{a}^{N_2}$	524
15.5.5	Upstream-Bias Functions α , δ , α^N	526
15.6	Polar Variation of Upstream-Bias	527
15.7	Incompressible-Flow Formulation	529
15.8	Variational Statement and Boundary Conditions	532

15.9	Finite Element Galerkin Weak Statement	536
15.10	Implicit Runge-Kutta Time Integration	540
15.11	Computational Performance	541
15.11.1	Backward-Facing Step Diffuser Flow	542
15.11.2	Natural Convection	549
15.11.3	Lid-Driven Cavity Flow	556
15.11.4	Lid- and Buoyancy-Driven Cavity Flow	563
16	Multi-Dimensional Free-Surface Flows	571
16.1	Governing Equations	572
16.2	Gravity-Wave Equations	572
16.2.1	Streamline and Crossflow Components	573
16.2.2	Similarity Transformation	574
16.3	Characteristic Analysis and Velocity Components	575
16.3.1	Polar Variation of Characteristic Speeds	576
16.3.2	Regions of Supercritical and Subcritical Wave Propagation	578
16.3.3	Free-Surface Flow Characteristic Cones	578
16.3.4	Physical Multi-dimensional Upstream Bias	582
16.4	Gravity-Wave-Convection Flux Divergence Decomposition	584
16.4.1	Convection and Pressure-Gradient Components	584
16.4.2	Gravity-Wave Components	586
16.4.3	Combination of Flux Divergence Decompositions	588
16.5	Characteristics-Bias Flux-Divergence	590
16.5.1	Consistent Multi-Dimensional and Infinite Directional Upstream Bias	591
16.5.2	Upstream-Bias Eigenvalues and Oblique-Jump Capturing	591
16.5.3	Conditions on Upstream-Bias Functions and Eigenvalues	592
16.5.4	Upstream-Bias Functions α , α , and δ	593
16.5.5	Crossflow Upstream Function α^N	593
16.5.6	Variations of α , δ , α^N	595
16.6	Polar Variation of Upstream-Bias	596
16.7	Variational Statement and Boundary Conditions	598
16.7.1	Characteristics-Bias Euler and Navier-Stokes Systems	599
16.7.2	Weak Statement and Boundary Conditions	600
16.8	Finite Element Galerkin Weak Statement	605
16.9	Implicit Runge-Kutta Time Integration	607
16.10	Computational Performance	608
16.10.1	Asymmetric Dam-Break Flow	609
16.10.2	Backward-Facing Step Diffuser Flow	614
16.10.3	Stream-Driven Inlet Flow	620
16.10.4	$Fr=2.5$ Hydraulic-Jump Reflection	628
16.10.5	$Fr=5.0$ Supercritical Flow	631

17 Multi-Dimensional Compressible Flows	635
17.1 Euler and Navier-Stokes Systems	636
17.1.1 2-D Formulation	637
17.1.2 3-D Formulation	637
17.1.3 Dependent Variables	638
17.1.4 Constitutive Relations	638
17.1.5 Equations of State and Speed of Sound	638
17.2 Acoustics Equations and Streamline and Crossflow Components	639
17.2.1 Acoustics Systems	640
17.2.2 Streamline and Crossflow Components	643
17.2.3 Similarity Transformation	646
17.3 Characteristic Analysis and Velocity Components	648
17.3.1 Characteristic Velocity Components	648
17.3.2 Polar Variation of Characteristic Speeds	650
17.3.3 Regions of Supersonic and Subsonic Wave Propagation	653
17.4 Acoustics-Convection Euler Flux Divergence Decomposition	655
17.4.1 2-D Convection and Pressure-Gradient Components	656
17.4.2 2-D Acoustic Components	657
17.4.3 Combination of 2-D Acoustics-Convection Characteristics-Bias Decompositions	660
17.4.4 3-D Convection and Pressure-Gradient Components	662
17.4.5 3-D Acoustic Components	663
17.4.6 Combination of 3-D Acoustics-Convection Characteristics-Bias Decompositions	666
17.5 Characteristics-Bias Euler Flux Divergence	668
17.5.1 Consistent Multi-Dimensional and Infinite-Directional Upstream Bias	670
17.5.2 Upstream-Bias Eigenvalues and Oblique-Shock Capturing	671
17.5.3 Conditions on Upstream-Bias Functions and Eigenvalues	672
17.5.4 Upstream-Bias Functions α , α , and δ	673
17.5.5 Crossflow Upstream-Bias Function α^N	675
17.5.6 Variations of α , δ , α^N	677
17.6 Polar Variation of Upstream-Bias	678
17.7 Variational Statement and Boundary Conditions	681
17.7.1 Characteristics-Bias Euler and Navier-Stokes Systems	681
17.7.2 Weak Statement and Boundary Conditions	684
17.8 Finite Element Galerkin Weak Statement	688
17.9 Implicit Runge-Kutta Time Integration	691
17.10 Computational Results	692
17.10.1 Hypersonic Blunt-Body Flow	692
17.10.2 Airfoil Flows	696
17.10.3 Inlet Flows	702
17.10.4 Convergence to Steady State	709
17.11 Computational Performance	709