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# Noise-Induced Transitions

Theory and Applications  
in Physics, Chemistry, and Biology

With 56 Figures

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