

Contents

Foreword — XIII

1	Sets and functions — 1
2	Numbers — 7
2.1	Rational numbers — 7
2.2	Whole numbers — 11
2.3	Real numbers — 14
2.3.1	Construction of the reals I — 14
2.3.2	Newton's method — 16
2.3.3	Construction of the reals II — 17
2.3.4	Metric spaces and completeness — 22
2.3.5	Construction of the reals III — 23
2.4	Complex numbers — 25
3	Vectors and their spaces — 28
3.1	Linear independence and bases — 30
3.2	Linear transformations — 33
3.3	The length of vectors — 39
3.3.1	Banach and Hilbert spaces — 43
3.3.2	Gram–Schmidt orthogonalization algorithm — 45
3.4	Coordinates and coordinate functions — 47
4	The topology of metric spaces — 49
4.1	Open sets — 49
4.1.1	Open sets and continuity — 50
4.2	Compactness and sequential compactness — 51
4.3	Extension of uniformly continuous maps — 53
4.4	Finite versus infinite-dimensional spaces — 54
4.4.1	Compactness — 54
4.4.2	Linearity and continuity — 58
4.5	Topology and convergence — 59
5	Differentiation — 62
5.1	Introduction — 62
5.2	Derivatives as linear maps — 65
5.3	Derivatives along curves — 68
5.4	Spaces of differentiable functions — 69
5.5	The inverse and implicit function theorems — 70
5.5.1	Banach fixed-point theorem — 71

5.5.2	The inverse function theorem —	72
5.5.3	Newton's method revisited —	75
5.5.4	The implicit function theorem —	75
5.6	A dive into coordinates —	76
6	Manifolds —	80
6.1	A motivating example —	80
6.2	Submanifolds of $\mathbb{R}^n$ —	81
6.3	Abstract manifolds —	85
6.3.1	The differential —	90
6.4	Concluding remarks —	91
7	Ordinary differential equations —	93
7.1	Linear systems of ODEs —	95
7.2	Existence and uniqueness for nonlinear ODEs —	100
7.3	Gradient flows —	103
7.4	Concluding remarks —	105
8	Optimization —	106
8.1	Singular value decomposition —	108
8.2	Eigenvalues of symmetric matrices —	111
8.2.1	The conjugate gradient method —	114
8.3	Convexity —	118
9	Partial differential equations —	127
9.1	The periodic heat equation —	127
9.2	The heat equation on the whole space —	135
9.3	Concluding remarks —	145
9.3.1	Infinite-dimensional optimization —	145
9.3.2	The spectrum of a linear operator —	146
10	Probability —	150
10.1	Axioms of probability —	150
10.2	Conditional probability and independence —	152
10.3	Random variables —	154
10.4	A discrete random walk and the diffusion equation —	158
10.5	Expected value of a random variable —	161
10.6	The gambler's ruin problem —	162
10.6.1	Winning probability —	163
10.6.2	A pool of games —	164
10.6.3	Game duration —	167
10.7	Vector-valued random variables and variance —	170

10.7.1	Principal component analysis —	171
10.8	Concluding remarks —	173
11	Answers to selected questions —	174
11.1	Answer to Chapter 1 question —	174
11.2	Answers to Chapter 2 questions —	174
11.3	Answers to Chapter 3 questions —	178
11.4	Answers to Chapter 4 questions —	185
11.5	Answers to Chapter 5 questions —	188
11.6	Answers to Chapter 6 questions —	190
11.7	Answers to Chapter 7 questions —	192
11.8	Answers to Chapter 8 questions —	193
11.9	Answers to Chapter 9 questions —	194
11.10	Answers to Chapter 10 questions —	196
Index —		199