## **Contents**

Preface v			vi		
No	otation	ns	X		
St	andar	d abbreviations	xiv		
I	Intr	oduction to stochastic analysis	1		
1	Tool	s from probability and analysis	3		
	1.1	Essentials of measure and probability	3		
	1.2	Characteristic functions	13		
	1.3	Conditioning	16		
	1.4	Infinitely divisible and stable distributions	21		
	1.5	Stable laws as the Holtzmark distributions	27		
	1.6	Unimodality of probability laws	29		
	1.7	Compactness for function spaces and measures	35		
	1.8	Fractional derivatives and pseudo-differential operators	42		
	1.9	Propagators and semigroups	48		
2	Brownian motion (BM)				
	2.1	Random processes: basic notions	58		
	2.2	Definition and basic properties of BM	62		
	2.3	Construction via broken-line approximation	66		
	2.4	Construction via Hilbert-space methods	69		
	2.5	Construction via Kolmogorov's continuity	71		
	2.6	Construction via random walks and tightness	73		
	2.7	Simplest applications of martingales	76		
	2.8	Skorohod embedding and the invariance principle	78		
	2.9	More advanced Hilbert space methods: Wiener chaos and stochastic			
		integral	81		
	2.10	Fock spaces, Hermite polynomials and Malliavin calculus	87		
	2.11	Stationarity: OU processes and Holtzmark fields	91		
3	Mar	Markov processes and martingales			
	3.1	Definition of Lévy processes	94		
	3.2	Poisson processes and integrals	96		



xvi Contents

	3.3	Construction of Lévy processes	103
	3.4	Subordinators	108
	3.5	Markov processes, semigroups and propagators	110
	3.6	Feller processes and conditionally positive operators	115
	3.7	Diffusions and jump-type Markov processes	125
	3.8	Markov processes on quotient spaces and reflections	130
	3.9	Martingales	132
	3.10	Stopping times and optional sampling	138
	3.11	Strong Markov property; diffusions as Feller processes with continu-	
		ous paths	143
	3.12	Reflection principle and passage times	147
4	SDE	, $\Psi  extbf{DE}$ and martingale problems	152
	4.1	Markov semigroups and evolution equations	152
	4.2	The Dirichlet problem for diffusion operators	158
	4.3	The stationary Feynman–Kac formula	162
	4.4	Diffusions with variable drift, Ornstein-Uhlenbeck processes	165
	4.5	Stochastic integrals and SDE based on Lévy processes	167
	4.6	Markov property and regularity of solutions	172
	4.7	Stochastic integrals and quadratic variation for square-integrable mar-	
		tingales	178
	4.8	Convergence of processes and semigroups	187
	4.9	Weak convergence of martingales	193
	4.10	Martingale problems and Markov processes	195
	4.11	Stopping and localization	199
II	Ma	rkov processes and beyond	203
5	Droo	esses in Euclidean spaces	204
3	5.1	Direct analysis of regularity and well-posedness	
	5.2	Introduction to sensitivity analysis	
	5.3	The Lie–Trotter type limits and T-products	
	5.4	Martingale problems for Lévy type generators: existence	
	5.5	Martingale problems for Lévy type generators: moments	
	5.6	Martingale problems for Lévy type generators: unbounded coefficients	
	5.7	• . • • • • • • • • • • • • • • • • • •	231
	5.8		240
	5.9	Stochastic monotonicity and duality	
		biochastic monotonicity and duanty	200
		Stochastic scattering	255
	5.10	Stochastic scattering	255
	5.10	Stochastic scattering	

Contents xvii

	5.12	Comments	262	
6	Processes in domains with a boundary			
	6.1	Stopped processes and boundary points	270	
	6.2	Dirichlet problem and mixed initial-boundary problem		
	6.3	The method of Lyapunov functions		
	6.4	Local criteria for boundary points		
	6.5	Decomposable generators in $\mathbb{R}^d_+$		
	6.6	Gluing boundary		
	6.7	Processes on the half-line	292	
	6.8	Generators of reflected processes	293	
	6.9	Application to interacting particles: stochastic LLN		
	6.10	Application to evolutionary games	304	
	6.11	Application to finances: barrier options, credit derivatives, etc	307	
	6.12	Comments	308	
7	Heat	kernels for stable-like processes	310	
	7.1	One-dimensional stable laws: asymptotic expansions		
	7.2	Stable laws: asymptotic expansions and identities		
	7.3	Stable laws: bounds		
	7.4	Stable laws: auxiliary convolution estimates		
	7.5	Stable-like processes: heat kernel estimates		
	7.6	Stable-like processes: Feller property		
	7.7	Application to sample-path properties		
	7.8	Application to stochastic control		
	7.9	Application to Langevin equations driven by a stable noise		
	7.10	Comments	348	
8	CTR	W and fractional dynamics	351	
	8.1	Convergence of Markov semigroups and processes	351	
	8.2	Diffusive approximations for random walks and CLT	354	
	8.3	Stable-like limits for position-dependent random walks	355	
	8.4	Subordination by hitting times and generalized fractional evolutions .	361	
	8.5	Limit theorems for position dependent CTRW	369	
	8.6	Comments	371	
9	Complex Markov chains and Feynman integral 37			
	9.1	Infinitely-divisible complex distributions and complex Markov chains	373	
	9.2	Path integral and perturbation theory	380	
	9.3	Extensions	385	
	9.4	Regularization of the Schrödinger equation by complex time or mass,		
		or continuous observation	390	

xviii	Content

9.5	Singular and growing potentials, magnetic fields and curvilinear state spaces	393		
9.6	Fock-space representation			
9.7	Comments	400		
Bibliography				
Index				