

Contents

Acknowledgments — V

1 Introduction — 1

1.1 Outline — 2

2 The linearity of the Mitchell order — 7

2.1 Introduction — 7

2.1.1 Normal ultrafilters and the Mitchell order — 7

2.1.2 The ultrapower axiom — 10

2.1.3 Outline of Chapter 2 — 10

2.2 Preliminary definitions — 11

2.2.1 Inner models — 11

2.2.2 Ultrapowers — 12

2.2.3 Close embeddings — 16

2.2.4 Uniform ultrafilters — 18

2.2.5 The Mitchell order on normal ultrafilters — 20

2.3 The linearity of the Mitchell order — 20

2.3.1 Weak comparison — 21

2.3.2 Weak comparison and the Mitchell order — 23

2.3.3 Weak comparison and the ultrapower axiom — 26

2.3.4 The ultrapower axiom and the Mitchell order — 29

2.3.5 Technical lemmas related to weak comparison — 31

3 The Ketonen order — 35

3.1 Introduction — 35

3.1.1 Ketonen's order — 35

3.1.2 Outline of Chapter 3 — 35

3.2 Preliminary definitions — 37

3.2.1 Fine ultrafilters on ordinals — 37

3.2.2 Limits of ultrafilters — 39

3.3 The Ketonen order — 41

3.3.1 Characterizations of the Ketonen order — 41

3.3.2 Basic properties of the Ketonen order — 44

3.3.3 The global Ketonen order — 48

3.4 Orders on ultrafilters — 50

3.4.1 The Mitchell order — 50

3.4.2 The Rudin–Keisler order — 51

3.4.3 The Lipschitz order — 56

3.4.4 The Ketonen order on filters — 60

3.5 The linearity of the Ketonen order — 63

4	The generalized Mitchell order — 71
4.1	Introduction — 71
4.1.1	The linearity of the generalized Mitchell order — 71
4.1.2	Outline of Chapter 4 — 71
4.2	Folklore of the generalized Mitchell order — 72
4.2.1	Strength and the Mitchell order — 72
4.2.2	Supercompactness and the Mitchell order — 76
4.2.3	The Kunen inconsistency — 81
4.2.4	The well foundedness of the generalized Mitchell order — 83
4.2.5	The nonlinearity of the generalized Mitchell order — 85
4.3	Dodd soundness — 87
4.3.1	Introduction — 87
4.3.2	Dodd sound embeddings, extenders, and ultrafilters — 87
4.3.3	The generalized Mitchell order on Dodd sound ultrafilters — 95
4.4	Generalizations of normality — 98
4.4.1	Normal fine ultrafilters — 100
4.4.2	Weakly normal ultrafilters — 102
4.4.3	Solovay's lemma — 106
4.4.4	Supercompactness and singular cardinals — 109
5	The Rudin–Frolík order — 117
5.1	Introduction — 117
5.1.1	Ultrafilters on the least measurable cardinal — 117
5.1.2	Outline of Chapter 5 — 118
5.2	The Rudin–Frolík order — 119
5.3	Below the first μ -measurable cardinal — 125
5.3.1	Introduction — 125
5.3.2	Irreducible ultrafilters and μ -measurability — 126
5.3.3	Factorization into irreducibles — 129
5.4	The structure of the Rudin–Frolík order — 135
5.4.1	The local ascending chain condition — 135
5.4.2	Pushouts and the Rudin–Frolík lattice — 136
5.4.3	The finiteness of the Rudin–Frolík order — 142
5.4.4	Translations and limits — 147
5.5	The internal relation — 150
5.5.1	A generalized Mitchell order — 150
5.5.2	The Mitchell order versus the internal relation — 151
5.5.3	Basic theory of the internal relation — 153
5.5.4	Commuting ultrapowers and wellfoundedness — 157
6	$V = \text{HOD}$ and GCH from UA — 163
6.1	Introduction — 163

6.1.1	The universe above a supercompact cardinal —	163
6.1.2	Outline of Chapter 6 —	163
6.2	Ordinal definability —	164
6.3	The generalized continuum hypothesis —	169
6.3.1	Introduction —	169
6.3.2	The number of supercompactness measures —	169
6.3.3	The local capturing property —	171
6.3.4	λ -Mitchell ultrafilters —	172
6.3.5	λ -Mitchell ultrafilters from UA —	174
6.3.6	GCH from UA —	177
6.3.7	$\diamond$ on the critical cofinality —	178
6.3.8	The size and saturation of the Vopěnka algebra —	179
7	The least supercompact cardinal —	183
7.1	Introduction —	183
7.1.1	The identity crisis —	183
7.1.2	Outline of Chapter 7 —	184
7.2	Strong compactness —	185
7.2.1	Some characterizations of strong compactness —	185
7.2.2	Ketonen's theorem —	188
7.2.3	Solovay's theorem —	190
7.3	The least ultrafilter $\mathcal{K}_\lambda$ —	192
7.3.1	Fréchet cardinals —	192
7.3.2	Ketonen ultrafilters —	194
7.3.3	Introducing $\mathcal{K}_\lambda$ —	196
7.3.4	The universal property of $\mathcal{K}_\lambda$ —	197
7.3.5	Independent families and the Hamkins properties —	200
7.3.6	The strength and supercompactness of $\mathcal{K}_\lambda$ —	205
7.4	Fréchet cardinals —	208
7.4.1	The next Fréchet cardinal —	208
7.4.2	The strong compactness of κ_λ —	211
7.4.3	The least supercompact cardinal —	215
7.4.4	The number of countably complete ultrafilters —	217
7.5	Isolation —	220
7.5.1	Isolated measurable cardinals —	220
7.5.2	Ultrafilters on an isolated cardinal —	223
7.5.3	Isolated cardinals and the GCH —	232
7.5.4	The linearity of the Mitchell order without GCH —	242
8	Higher supercompactness —	249
8.1	Introduction —	249
8.1.1	Obstructions to the supercompactness analysis —	249

8.1.2	Menas' theorem —	249
8.1.3	Complete UA —	251
8.1.4	Irreducible ultrafilters and supercompactness —	252
8.1.5	Outline of Chapter 8 —	254
8.2	The irreducibility theorem —	255
8.2.1	Tightness and irreducibility —	255
8.2.2	Translations of $\mathcal{K}_\lambda$ —	257
8.2.3	Elementary embeddings and normal filters —	259
8.2.4	A proof of the irreducibility theorem —	262
8.3	Resolving the identity crisis —	269
8.3.1	The equivalence of strong compactness and supercompactness —	270
8.3.2	Level-by-level equivalence at singular cardinals —	275
8.3.3	The Mitchell order, the internal relation and coherence —	280
8.4	Very large cardinals —	289
8.4.1	Huge cardinals —	289
8.4.2	Cardinal preserving elementary embeddings —	293
8.4.3	Supercompactness at inaccessible cardinals —	298

9 Open questions — 307

9.2.2	Weak comparison —	307
9.2.3	The ultrapower axiom —	307
9.3.1	The Ketonen order —	308
9.3.2	The Lipschitz order —	309
9.4.1	The generalized Mitchell order —	310
9.5.1	The Rudin–Frolík order —	311
9.5.2	The internal relation —	312
9.5.3	$V = \text{HOD}$ —	312
9.5.4	The generalized continuum hypothesis —	313
9.5.5	The ground axiom —	314
9.6.1	Isolated cardinals —	314
9.6.2	Local supercompactness —	315
9.7.1	The complete ultrapower axiom —	316
9.7.2	Cardinal preserving embeddings —	316
9.7.3	Supercompactness at inaccessible cardinals —	317

Bibliography — 319

Index — 321