

Contents

Acknowledgments — VII

Foreword — IX

1 Geometric motivation — 1

- 1.1 Bits from intersection theory — 1
- 1.2 Correspondences and incidence varieties — 4
 - 1.2.1 Examples — 4
 - 1.2.2 Algebraic counterpart — 6
- 1.3 Ruled joins — 7
- 1.4 Exercises — 10

2 Graded algebras — 13

- 2.1 Arbitrary gradings — 13
- 2.2 Notable bigraded algebras — 16
 - 2.2.1 The symmetric algebra — 16
 - 2.2.2 Other bigraded algebras — 18
- 2.3 Standard graded algebras — 18
 - 2.3.1 Rees algebras of ideals — 18
 - 2.3.2 Rees algebras of modules — 28
 - 2.3.3 Complements on the fiber cone — 36
 - 2.3.4 Cohomological invariants — 39
- 2.4 Exercises — 41

3 Rational maps, I — 47

- 3.1 Basics of rational maps — 47
 - 3.1.1 Terminology — 47
 - 3.1.2 Basic properties — 48
- 3.2 A characteristic-free criterion of birationality — 50
 - 3.2.1 Prolegomena: the Koszul–Hilbert lemma — 50
 - 3.2.2 Preliminaries on rational maps on reduced varieties — 51
 - 3.2.3 The Jacobian dual rank versus birationality — 59
 - 3.2.4 Statement and proof of the criterion — 63
 - 3.2.5 A role of syzygies — 67
- 3.3 Birational combinatorics — 70
 - 3.3.1 The underlying arithmetic — 71
 - 3.3.2 Monomial birational maps in degree two — 73
- 3.4 On the degree of a rational map — 77
 - 3.4.1 Basic ingredients — 77
 - 3.4.2 Degree upper bounds — 81

3.5 Exercises — 85

4 Rational maps, II — 89

- 4.1 Complements on the base ideal — 89**
 - 4.1.1 Degree sequences — 89**
 - 4.1.2 The semilinear relation type — 92**
- 4.2 Inversion factors of birational maps — 93**
 - 4.2.1 Basic landscape — 94**
 - 4.2.2 Relation to symbolic powers — 96**
- 4.3 Newton complementary dual — 97**
 - 4.3.1 Preliminaries — 97**
 - 4.3.2 Main results — 99**
 - 4.3.3 Relation to the graph of a rational map — 104**
- 4.4 Elements of rational maps over a ground ring — 108**
 - 4.4.1 Algebraic data — 108**
 - 4.4.2 Rational maps and representatives — 110**
 - 4.4.3 Degree under specialization — 113**
- 4.5 Exercises — 115**

5 The gradient ideal — 119

- 5.1 The underlying geometry — 119**
 - 5.1.1 Linear type — 120**
 - 5.1.2 Hyperplane arrangements — 121**
 - 5.1.3 Behavior along a family — 129**
- 5.2 Free divisors — 140**
 - 5.2.1 General nonsense — 140**
 - 5.2.2 Algebraic logarithmic vector fields — 141**
 - 5.2.3 Families of free divisors — 145**
 - 5.2.4 Weighted homogeneous divisors — 148**
- 5.3 Exercises — 155**

6 Cremona transformations, I — 159

- 6.1 Plane Cremona transformations: homaloidal nets — 159**
 - 6.1.1 Weighted cluster of base points — 159**
 - 6.1.2 Relation to fat ideals — 163**
 - 6.1.3 From fat ideals back to the Cremona base ideal — 172**
 - 6.1.4 Proper homaloidal types with large highest virtual multiplicity — 173**
- 6.2 Plane Cremona transformations: subhomaloidal types — 179**
 - 6.2.1 Arithmetic quadratic transformations acting on fat points — 179**
 - 6.2.2 Subhomaloidal multiplicity sets — 181**
 - 6.2.3 3-uniform subhomaloidal types — 183**
 - 6.2.4 Main theorem — 199**

6.2.5	Relation to Bordiga–White parametrizations —	203
6.3	Exercises —	206
7	Cremona transformations, II —	209
7.1	Generalized de Jonquières transformations —	209
7.1.1	Fully downgraded sequences —	211
7.1.2	Generalized de Jonquières transformations —	213
7.1.3	Generalized de Jonquières transformations under Newton duality —	215
7.1.4	Generalized de Jonquières transformations with identity support —	218
7.1.5	The minimal free resolution of the base ideal —	218
7.2	The graph (blowup) of a Cremona map —	220
7.2.1	The graph of a de Jonquières map with identity support —	220
7.2.2	Cremona maps with a Cohen–Macaulay graph —	227
7.2.3	Cremona maps and symbolic Rees algebras —	231
7.3	Monomial Cremona transformations —	235
7.3.1	The group of monomial Cremona transformations —	235
7.3.2	Monomial Cremona transformations of degree 2 —	243
7.3.3	A duality principle —	256
7.4	Exercises —	259
8	Steps in elimination —	263
8.1	Elimination algebras —	263
8.2	Implicitization —	266
8.3	De Jonquières like implicitization —	267
8.3.1	Algebraic prelims —	267
8.3.2	The (If, g) -parametrization —	268
8.3.3	The equations of the Rees algebra —	273
8.4	Implicitization in dimension ≤ 1 —	278
8.4.1	Dimension zero —	278
8.4.2	Dimension one —	293
8.5	The classical resultant —	295
8.6	Eliminating from correspondences —	297
8.7	Exercises —	300
9	The Gauss map —	303
9.1	Gauss image of higher order —	303
9.1.1	Notable cases. I: the ordinary Gauss image —	305
9.1.2	Notable cases. II: the dual variety —	306
9.1.3	Relation to the tangential variety —	308
9.2	The Gauss image of a unirational variety —	310

9.2.1	The algebraic structural data —	310
9.2.2	The Gauss image of toric varieties —	313
9.2.3	Algebras of Veronese type —	314
9.3	Abstract Gauss maps —	316
9.4	The geometry of the analytic spread —	317
9.4.1	Maximal analytic spread —	317
9.4.2	Analytic spread versus the Fitting ideal —	322
9.4.3	Case study: the module of differentials —	323
9.4.4	Main theorem of abstract Gauss maps —	325
9.5	Exercises —	329
10	Tangent varieties —	333
10.1	Algebraic Fulton–Hansen theorems —	333
10.1.1	Algebraic preliminaries —	333
10.1.2	The underlying geometry —	337
10.1.3	The tangential variety to a unirational variety —	340
10.2	Refined properties of tangent algebras —	341
10.2.1	Expected star dimension —	343
10.2.2	Set theoretic starlike linearity —	347
10.2.3	Starlike linearity —	348
10.2.4	Behavior under flat deformation —	351
10.2.5	A glimpse of the blowup of the Kähler differentials —	354
10.3	Exercises —	356
11	Embedded joins and secants —	359
11.1	Joins —	359
11.1.1	Algebraic side —	359
11.1.2	The geometric background —	360
11.1.3	Relation to the Bayer–Mumford deformation —	363
11.1.4	The embedded join of a union of linear spaces —	366
11.1.5	Initial degree of a join —	368
11.2	Secants —	369
11.2.1	Secants of determinantal and Pfaffian schemes —	370
11.2.2	Secants of varieties cut by quadrics —	374
11.2.3	Secant algebras of monomial ideals —	377
11.2.4	Relation to starlike linearity —	378
11.2.5	Higher secants —	380
11.3	Excerpts on the diagonal ideal —	381
11.3.1	Syzygies and dimension —	382
11.3.2	The linear type property —	383
11.4	Exercises —	386

12	Aluffi algebras — 389
12.1	Relation to Chern classes — 389
12.2	The case of ideals: embedded version — 390
12.2.1	Preliminaries — 390
12.2.2	Dimension — 392
12.2.3	Torsion and minimal primes — 395
12.2.4	Relation to the Artin–Rees number — 398
12.2.5	Selected examples — 402
12.2.6	The Aluffi algebra of a gradient ideal — 405
12.3	The case of modules — 407
12.3.1	The embedded version — 408
12.3.2	Taking inverse limit — 409
12.3.3	The module of derivations of a polynomial form — 413
12.3.4	The normal module of a perfect ideal of codimension 2 — 421
12.4	Exercises — 427

Bibliography — 431

Index — 441