

Contents

Preface — VII

1	Ring theoretical foundations — 1
1.1	Algebras, actions and modules: definition of the basic objects — 1
1.1.1	Algebras — 1
1.1.2	Modules — 4
1.1.3	Direct sums and products of modules — 7
1.1.4	Group algebras and modules over a group — 9
1.2	Maschke, Wedderburn and Krull-Schmidt — 13
1.2.1	Maschke's theorem — 16
1.2.2	The Krull-Schmidt theorem — 19
1.2.3	Wedderburn's structure theorem on semisimple artinian rings — 25
1.3	Group rings as semisimple algebras — 35
1.4	Exercises — 38
2	Characters — 45
2.1	Definitions and basic properties — 45
2.2	Class functions, linking characters to the semisimple group algebra — 47
2.3	Multiplicities of simple submodules; orthogonality relations — 51
2.4	Character degrees, Burnside's theorem and the Dixon-Schneider algorithm — 58
2.4.1	Integral elements in rings, rings of integers — 58
2.4.2	Group rings and integrality; character degrees — 60
2.4.3	Burnside's $p^a q^b$ -theorem — 62
2.4.4	The mathematical principle of the Dixon-Schneider algorithm — 67
2.5	Application to number theory; Dirichlet's theorem on arithmetic progressions — 73
2.6	Exercises — 84
3	Tensor products, Mackey formulas and Clifford theory — 93
3.1	The tensor product — 93
3.1.1	The definition of a tensor product — 93
3.1.2	Homomorphisms and tensor product — 96
3.1.3	The case of tensor products over fields — 99
3.1.4	Additional structure, algebras and modules — 100
3.2	Using tensor products, induced modules — 104
3.3	Mackey's formula — 112
3.4	Clifford theory — 119
3.4.1	Inertia group — 120

3.4.2	Factor sets, second group cohomology —	121
3.4.3	Clifford's main theorem —	126
3.4.4	Small and big inertia groups: Blichfeldt's and Ito's theorem —	129
3.5	Exercises —	135
4	Bilinear forms on modules —	139
4.1	Invariant bilinear forms —	139
4.2	The Frobenius Schur indicator —	143
4.3	Quadratic modules in characteristic 2 —	145
4.3.1	First group cohomology —	146
4.3.2	Properties of symmetric and exterior products —	149
4.3.3	Quadratic modules and group cohomology —	151
4.4	Exercises —	152
5	Brauer induction, Brauer's splitting field theorem —	159
5.1	Grothendieck and character ring —	159
5.2	Brauer induction formula —	166
5.3	Brauer's splitting field theorem —	173
5.4	The semisimple algebra case —	177
5.5	Exercises —	181
6	Some homological algebra methods in ring theory —	183
6.1	Some facts about Noetherian and artinian modules —	183
6.1.1	Elementary properties and definitions —	183
6.1.2	Composition series —	187
6.2	On projective modules —	189
6.3	Extension groups —	195
6.3.1	Degree 1 via syzygies —	196
6.3.2	Higher <i>Ext</i> -groups and applications —	199
6.4	Hereditary algebras —	209
6.5	Flatness —	212
6.6	Exercises —	218
7	Some algebraic number theory —	223
7.1	Some supplements on algebraic integers —	223
7.2	Primary decomposition —	225
7.3	Discrete valuation rings —	227
7.3.1	Ideal structure of discrete valuation rings —	227
7.3.2	Valuations —	228
7.3.3	Completions —	231
7.3.4	Extension of valuations —	236
7.4	Fractional ideals —	239

7.5	Dedekind domains —	242
7.6	The strong approximation theorem —	255
7.7	Exercises —	261
8	Some notions of integral representations —	265
8.1	Classical orders and their lattices —	266
8.1.1	Basic definitions and examples —	266
8.1.2	Orders as pullbacks —	270
8.1.3	Orders, lattices and localisation —	273
8.2	Reduced norms, traces, characteristic polynomials —	275
8.3	Maximal orders —	280
8.3.1	Definition, existence and examples —	280
8.3.2	Maximal orders are hereditary: the Auslander Buchsbaum theorem —	289
8.4	The Jordan Zassenhaus theorem —	295
8.5	Class groups for orders —	301
8.5.1	Definitions and elementary properties —	301
8.5.2	Idèle class groups —	305
8.6	Swan's example —	313
8.7	Galois module structure —	328
8.7.1	Ramifications —	328
8.7.2	Taylor's results on Galois module structure —	330
8.8	Exercises —	333
9	Solution to selected exercises —	339
	Bibliography —	349
	Index —	353