

# Contents

## Preface — VII

## Preface to the second edition — XI

### 1 Introduction — 1

- 1.1 Gyroscopic stabilization on a rotating surface — 1
  - 1.1.1 Brouwer's mechanical model — 1
  - 1.1.2 Eigenvalue problems and the characteristic equation — 2
  - 1.1.3 Eigencurves and bifurcation of multiple eigenvalues — 4
  - 1.1.4 Singular stability boundary of the rotating saddle trap — 8
- 1.2 Manifestations of Brouwer's model in physics — 10
  - 1.2.1 Stability of deformable rotors — 10
  - 1.2.2 Foucault's pendulum, Bryan's effect, Coriolis vibratory gyroscopes, and the Hannay–Berry phase — 15
  - 1.2.3 Polarized light within a cholesteric liquid crystal — 17
  - 1.2.4 Helical magnetic quadrupole focussing systems — 18
  - 1.2.5 Ponderomotive magnetism and the Cox–Levi theorem — 20
  - 1.2.6 Stability of Lagrange's triangular libration points  $L_4$  and  $L_5$  — 23
  - 1.2.7 Modulational instability — 26
- 1.3 Brouwer's problem with damping and circulatory forces — 31
  - 1.3.1 Circulatory forces — 32
  - 1.3.2 Dissipation-induced instability of negative energy modes — 33
  - 1.3.3 Circulatory systems and the destabilization paradox — 34
  - 1.3.4 Merkin's theorem, Nicolai's paradox, and subcritical flutter — 36
  - 1.3.5 Indefinite damping and parity-time ( $\mathcal{PT}$ ) symmetry — 37
  - 1.3.6  $\mathcal{PT}$ -symmetry, pseudo-Hermiticity, and G-Hamiltonian matrices — 41
  - 1.3.7 Heavy damping and high-order exceptional points — 42
- 1.4 Scope of the book — 44

### 2 Lyapunov stability and linear stability analysis — 47

- 2.1 Main facts and definitions — 48
  - 2.1.1 Stability, instability, and uniform stability — 48
  - 2.1.2 Attractivity and asymptotic stability — 49
  - 2.1.3 Autonomous, nonautonomous, and periodic systems — 50
- 2.2 The direct (second) method of Lyapunov — 50
  - 2.2.1 Lyapunov functions — 50
  - 2.2.2 Lyapunov and Persidskii theorems on stability — 51
  - 2.2.3 Chetaev and Lyapunov theorems on instability — 52

2.3	The indirect (first) method of Lyapunov —	53
2.3.1	Linearization —	53
2.3.2	The characteristic exponent of a solution —	54
2.3.3	Lyapunov regularity of linearization —	55
2.3.4	Stability and instability in the first approximation —	56
2.4	Linear stability analysis —	57
2.4.1	Autonomous systems —	57
2.4.2	Lyapunov transformation and reducibility —	58
2.4.3	Periodic systems —	60
2.4.4	Example. Coupled parametric oscillators —	61
2.5	Algebraic criteria for asymptotic stability —	64
2.5.1	Lyapunov's matrix equation and stability criterion —	64
2.5.2	The Leverrier–Faddeev algorithm and Lewin's formula —	65
2.5.3	Müller's solution to the matrix Lyapunov equation —	66
2.5.4	Inertia theorems and observability index —	67
2.5.5	Hermite's criterion via the matrix Lyapunov equation —	68
2.5.6	Routh–Hurwitz, Liénard–Chipart, Bilharz, and Jury criteria —	70
2.6	Robust Hurwitz stability vs. structural instability —	72
2.6.1	Multiple eigenvalues: singularities and structural instabilities —	73
2.6.2	Multiple eigenvalues: spectral abscissa minimization and robust stability —	75
2.6.3	Abscissa minimization and the self-stability of bicycles —	77
<b>3</b>	<b>Hamiltonian and gyroscopic systems —</b>	<b>84</b>
3.1	Sobolev's top and an indefinite metric —	85
3.2	Elements of Pontryagin and Krein space theory —	88
3.3	Canonical and Hamiltonian equations —	91
3.3.1	Krein signature of eigenvalues —	93
3.3.2	Krein collision or linear Hamiltonian–Hopf bifurcation —	94
3.3.3	MacKay's cones, veering, and instability bubbles —	95
3.3.4	Instability degree and count of eigenvalues —	97
3.3.5	Graphical interpretation of the Krein signature —	99
3.3.6	Strong stability: robustness to Hamiltonian's variation —	103
3.3.7	Inertia theorems and stability of gyroscopic systems —	104
3.3.8	Positive and negative energy modes and Krein signature —	105
3.3.9	Dispersive wave propagation in conservative systems —	107
3.3.10	Absolute and convective instability —	109
<b>4</b>	<b>Reversible and circulatory systems —</b>	<b>111</b>
4.1	Reversible systems —	111
4.2	Nonconservative positional forces —	112
4.3	Circulatory systems —	113

4.3.1	Divergence and flutter instabilities —	114
4.3.2	Multiple parameter families of circulatory systems —	114
4.3.3	Generic singularities on the stability boundary —	115
4.4	Perturbation of eigenvalues —	117
4.4.1	Simple eigenvalue —	118
4.4.2	Double eigenvalue of geometric multiplicity 1 —	119
4.4.3	Double eigenvalue of geometric multiplicity 2 —	121
4.4.4	Triple eigenvalue of geometric multiplicity 1 —	122
4.5	Geometry of the stability boundary —	124
4.5.1	Linear and quadratic approximations at smooth points —	124
4.5.2	Singularities in two-parameter circulatory systems —	126
4.5.3	Example. Stabilization of comfortable walking —	130
4.5.4	Singularities in three-parameter circulatory systems —	133
4.5.5	The cone $\alpha\alpha$ and Merkin's instability theorem —	141
4.5.6	Example: a brake disk in distributed frictional contact —	142
4.5.7	Example: stability of an airfoil in an inviscid flow —	145
4.6	Eigencurves, their crossing and veering —	149
4.6.1	Convex flutter domain: conical point $\alpha\alpha$ —	149
4.6.2	Convex/concave flutter domain: smooth points $\alpha^2$ —	150
4.7	Parametric optimization of circulatory systems —	154
4.7.1	Example: optimization of Ziegler's pendulum —	155
4.7.2	A nonsmooth and nonconvex optimization problem —	157
4.7.3	The gradient of the critical load —	158
4.7.4	An infinite gradient at the crossing of the eigencurves —	159
4.7.5	Improving variations and necessary conditions for optimality in the case where the eigencurves cross —	159
<b>5</b>	<b>Influence of structure of forces on stability —</b>	<b>162</b>
5.1	Undamped potential systems —	163
5.1.1	Lagrange's theorem and Poincaré instability degree —	163
5.1.2	Rayleigh's theorem on movement of eigenvalues —	164
5.1.3	Steady-state bifurcation —	164
5.2	Damped potential systems —	164
5.2.1	Overdamped and heavily damped systems —	165
5.2.2	Self-stable and heavily-damped TMS bicycle —	169
5.2.3	Indefinitely damped systems —	178
5.3	Undamped gyroscopic systems —	184
5.3.1	Extension of Rayleigh's theorem —	184
5.3.2	Criteria of gyroscopic stabilization —	185
5.4	Damped gyroscopic systems —	186
5.4.1	Kelvin–Tait–Chetaev theorem —	186
5.5	Circulatory systems with and without velocity-dependent forces —	187

5.5.1	Merkin's theorem and Bulatovic's flutter condition —	189
5.5.2	Bottema–Lakhadanov–Karapetyan theorem —	189
5.5.3	Stabilizing and destabilizing damping configurations —	190
<b>6</b>	<b>Dissipation-induced instabilities —</b>	<b>194</b>
6.1	Crandall's gyropendulum —	194
6.1.1	Conservative gyroscopic stabilization and its destruction by stationary damping —	195
6.1.2	Singular threshold of the nonconservative gyroscopic stabilization —	196
6.1.3	Imperfect Krein collision and exchange of instability between negative and positive energy modes —	197
6.2	Gyroscopic stabilization of nonconservative systems —	199
6.2.1	The case of $m = 2$ degrees of freedom —	200
6.2.2	The case of arbitrary even $m$ —	207
6.3	Near-Hamiltonian systems —	211
6.4	Gyroscopic and circulatory systems as limits of dissipative systems —	213
<b>7</b>	<b>Nonself-adjoint boundary eigenvalue problems for differential operators and operator matrices dependent on parameters —</b>	<b>222</b>
7.1	Adjoint boundary eigenvalue problems —	224
7.2	Perturbation of eigenvalues —	226
7.2.1	Semisimple eigenvalues —	227
7.2.2	Multiple eigenvalues with the Keldysh chain —	229
7.2.3	Higher order perturbation terms for double nonderogatory eigenvalues —	231
7.2.4	Degenerate splitting of double nonderogatory eigenvalues —	233
7.3	Example: a rotating circular string with an elastic restraint —	234
7.4	Example: the Herrmann–Smith paradox —	239
7.4.1	Formulation of the problem —	239
7.4.2	Stationary flutter domain and mobile divergence region —	242
7.4.3	Sensitivity of the critical flutter load to the redistribution of the elasticity modulus —	244
7.5	Example: Beck's column loaded by a partially follower force —	245
7.5.1	The stability-divergence boundary (point <i>A</i> ) —	247
7.5.2	The flutter threshold of Beck's column (point <i>C</i> ) —	248
7.5.3	The singularity $0^2$ on the stability boundary (point <i>B</i> ) —	252
<b>8</b>	<b>The destabilization paradox in continuous circulatory systems —</b>	<b>255</b>
8.1	Movement of eigenvalues under a velocity-dependent perturbation —	258

- 8.1.1 Generalized boundary eigenvalue problem — 259
- 8.1.2 Variation of parameters that is transversal to the stability boundary — 261
- 8.1.3 Variation of parameters that is tangential to the stability boundary — 262
- 8.1.4 Transfer of instability between modes — 264
- 8.1.5 Drop in the critical frequency — 266
- 8.2 Singular threshold of the flutter instability — 267
- 8.2.1 Drop in the critical flutter load — 267
- 8.2.2 The “no drop” condition and the tangent cone to the domain of asymptotic stability — 268
- 8.3 Example: dissipation-induced instability of Beck’s column — 271
- 8.3.1 Beck’s column without damping — 272
- 8.3.2 Beck’s column with Kelvin–Voigt and viscous damping — 273
- 8.3.3 Viscoelastic Beck’s column with a dash-pot — 278
- 8.3.4 Ziegler’s pendulum with a dash-pot — 281
- 8.4 Application to finite-dimensional systems — 283
- 8.4.1 The destabilization paradox in Ziegler’s pendulum — 284
- 8.5 Experimental detection of the Ziegler–Bottema destabilization paradox — 288
- 8.5.1 The destabilizing effect of external damping — 288
- 8.5.2 Ziegler–Bottema paradox due to vanishing external damping — 290
- 8.5.3 Ziegler–Bottema paradox for the Pflüger column with external damping — 297
- 8.5.4 The ‘flutter machine’: design, realization, and validation — 301
- 8.5.5 Experimental results versus theoretical and computational predictions — 307
- 9 The MHD kinematic mean field  $\alpha^2$ -dynamo — 314**
  - 9.1 Eigenvalue problem for  $\alpha^2$ -dynamo — 314
  - 9.2 Uniform  $\alpha$ -profiles generate only nonoscillatory dynamos — 318
  - 9.2.1 Conducting exterior: self-adjointness in a Krein space — 319
  - 9.2.2 Basis properties of eigenfunctions — 319
  - 9.2.3 Spectral mesh of eigencurves — 320
  - 9.2.4 Deformation of the spectral mesh via transition from conducting to insulating surrounding — 322
  - 9.3 Nonhomogeneous  $\alpha$ -profiles and the conducting exterior — 323
  - 9.3.1  $l \geq 0$ : definite Krein signature prohibits formation of complex eigenvalues — 324
  - 9.3.2  $l = 0$ : oscillating solutions from the repeated decaying modes with mixed Krein signature — 328

9.3.3	$l = 0$ : Fourier components of $\alpha(x)$ determine the unfolding pattern of the spectral mesh — 332
9.4	Insulating boundary conditions induce unstable oscillations — 335
9.4.1	$l = 0$ : complex unfolding of double eigenvalues with definite Krein signature — 337
<b>10</b>	<b>Campbell diagrams of gyroscopic continua and subcritical friction-induced flutter — 342</b>
10.1	Friction-induced vibrations and sound generation — 342
10.2	Example. Subcritical flutter of a rotating circular string — 345
10.3	Axially symmetric rotor with anisotropic stator — 353
10.3.1	Sensitivity analysis of the Campbell diagram — 355
10.3.2	MacKay's eigenvalue cones and instability bubbles — 357
10.3.3	Double-coffee-filter singularity near the crossings with definite Krein signature — 361
10.3.4	Unfolding MacKay's cones with mixed Krein signature — 364
10.3.5	Indefinite damping as a reason for subcritical flutter — 365
10.3.6	Destabilizing role of circulatory forces — 368
10.4	Example: eigenvalue surfaces of the rotating circular string — 371
10.5	How to play a disk brake? — 375
<b>11</b>	<b>Non-Hermitian perturbation of Hermitian matrices with physical applications — 377</b>
11.1	Eigenvalue movement through a 1 : 1 resonance in complex matrices — 380
11.1.1	Diabolical point (DP): passing of eigenvalues — 381
11.1.2	Exceptional point (EP): splitting of eigenvalues — 382
11.2	Eigensurfaces associated with DPs — 383
11.2.1	Complex perturbation of a Hermitian matrix family — 384
11.2.2	DP in the spectrum of real symmetric matrices — 385
11.2.3	How a DP unfolds into the conical wedge of Wallis — 385
11.2.4	Inflating the diabolical point into an exceptional ring — 389
11.2.5	Example: flutter instability in granular flow — 391
11.3	Unfolding conical singularities in crystal optics — 391
11.3.1	DPs in Hamilton's conical refraction — 393
11.3.2	Approximation of the dispersion surface near a DP — 395
11.3.3	Eigensurfaces of absorption- and chirality-dominated crystals — 396
11.4	Eigensurfaces associated with EPs — 398
11.5	Perturbation of eigenvectors and Berry phase — 403
11.5.1	Hermitian case: geometric phase around a DP — 403
11.5.2	Non-Hermitian case: geometric phase around an EP — 405
11.5.3	Geometric phase around an EP in a microwave cavity — 409

<b>12</b>	<b>Double-diffusive instabilities in hydro- and magnetohydrodynamics — 412</b>
12.1	Standard, azimuthal, and helical magnetorotational instability — 412
12.1.1	Cylindrical Couette–Taylor flow — 412
12.1.2	Paradox of Velikhov and Chandrasekhar — 415
12.1.3	Magnetorotational instability in astrophysics and its mechanical analogues — 416
12.1.4	Laboratory experiments with CT-flow in axial, azimuthal, and helical magnetic fields — 418
12.2	Mathematical setting — 420
12.2.1	Nonlinear equations and a steady state — 420
12.2.2	Linearization with respect to nonaxisymmetric perturbations — 422
12.3	Geometrical optics approximation — 422
12.3.1	Dispersion relation of AMRI — 425
12.3.2	Dispersion relation of HMRI — 426
12.3.3	The threshold of the standard MRI — 428
12.3.4	Singularities and the Velikhov–Chandrasekhar paradox — 430
12.3.5	The singular threshold of the HMRI and connection of HMRI and SMRI through a spectral exceptional point for magnetic Rossby number $R_b = -1$ — 433
12.4	Extending the range of HMRI/AMRI to Keplerian flows with arbitrary $R_b$ — 438
12.4.1	Inductionless approximation — 440
12.4.2	Extremal properties of the critical hydrodynamic Rossby number — 440
12.4.3	Continuation of the Liu limits to arbitrary $R_b$ — 442
12.4.4	Scaling law of the inductionless MRI — 444
12.4.5	Growth rates of HMRI and AMRI and the critical Reynolds number — 444
12.4.6	HMRI and AMRI as magnetically destabilized inertial waves — 446
12.4.7	AMRI as a dissipation-induced instability of Chandrasekhar’s equipartition solution — 448
12.4.8	Transition from AMRI to the Tayler instability — 450
12.5	AMRI as a double-diffusive instability — 452
12.5.1	G-Hamiltonian structure of the diffusionless AMRI — 453
12.5.2	Krein signature and splitting of double eigenvalues with Jordan block — 455
12.5.3	Neutral stability curves — 456
12.5.4	The Krein collision at the linear Hamilton–Hopf bifurcation threshold — 458
12.5.5	Dissipative perturbation of simple imaginary eigenvalues — 460
12.5.6	Weak ohmic diffusion destabilizes positive energy waves at low $P_m$ — 461

12.5.7      Diffusionless and double-diffusive criteria are connected at  
                  $P_m = 1$  — **462**

12.5.8      Double-diffusive instability at  $P_m \neq 1$  and arbitrary  $Re$  and  $Rm$  — **463**

12.6          Concluding remarks — **473**

**Bibliography — 479**

**Index — 519**