

Contents

- 1 Delay Differential Equations** 1
 - 1.1 Introduction 1
 - 1.1.1 DDE with Single Constant Delay 3
 - 1.1.2 DDE with Discrete Delays 4
 - 1.1.3 DDE with Distributed Delay 5
 - 1.1.4 DDE with State-Dependent Delay 6
 - 1.1.5 DDE with Time-Dependent Delay 6
 - 1.2 Constructing the Solution for DDEs with Single Constant Delay .. 7
 - 1.2.1 Linear Delay Differential Equation 8
 - 1.2.2 Numerical Simulation of DDEs 10
 - 1.2.3 Nonlinear Delay Differential Equations 11
 - 1.3 Salient Features of Chaotic Time-Delay Systems 13
 - References 13

- 2 Linear Stability and Bifurcation Analysis** 17
 - 2.1 Introduction 17
 - 2.2 Linear Stability Analysis 17
 - 2.2.1 Example: Linear Delay Differential Equation 19
 - 2.3 A Geometric Approach to Study Stability 20
 - 2.3.1 Example: Linear Delay Differential Equation 21
 - 2.4 A General Approach to Determine Linear Stability of Equilibrium Points 22
 - 2.4.1 Characteristic Equation 22
 - 2.4.2 Stability Conditions 22
 - 2.4.3 Stability Curves/Surfaces in the (τ, a, b) Parameter Space 23
 - 2.4.4 Extension to Coupled DDEs/Complex Scalar DDEs 24
 - 2.4.5 Bifurcation Analysis 25
 - 2.4.6 Results of Stability Analysis 25
 - 2.4.7 A Theorem on the Stability of Equilibrium Points 26
 - 2.4.8 Example: Linear Delay Differential Equation 26
 - References 29

3	Bifurcation and Chaos in Time-Delayed Piecewise Linear Dynamical System	31
3.1	Introduction	31
3.2	Simple Scalar First Order Piecewise Linear DDE	32
3.2.1	Fixed Points and Linear Stability	33
3.3	Numerical Study of the Single Scalar Piecewise Linear Time-Delay System	36
3.3.1	Dynamics in the Pseudospace	36
3.3.2	Transients	37
3.3.3	One and Two Parameter Bifurcation Diagrams	41
3.3.4	Lyapunov Exponents and Hyperchaotic Regimes	43
3.4	Experimental Realization using PSPICE Simulation	44
3.5	Stability Analysis and Chaotic Dynamics of Coupled DDEs	46
3.5.1	Fixed Points and Linear Stability	46
3.6	Numerical Analysis of the Coupled DDE	49
3.6.1	Transients	50
3.6.2	One and Two Parameter Bifurcation Diagrams	51
	References	53
4	A Few Other Interesting Chaotic Delay Differential Equations	55
4.1	Introduction	55
4.2	The Mackey-Glass System: A Typical Nonlinear DDE	55
4.2.1	Mackey-Glass Time-Delay System	55
4.2.2	Fixed Points and Linear Stability Analysis	56
4.2.3	Time-Delay $\tau = 0$	57
4.2.4	Time-Delay $\tau > 0$	57
4.2.5	Numerical Simulation: Bifurcations and Chaos	62
4.2.6	Experimental Realization Using Electronic Circuit	64
4.3	Other Interesting Scalar Chaotic Time-Delay Systems	67
4.3.1	A Simple Chaotic Delay Differential Equation	67
4.3.2	Ikeda Time-Delay System	67
4.3.3	Scalar Time-Delay System with Polynomial Nonlinearity	69
4.3.4	Scalar Time-Delay System with Other Piecewise Linear Nonlinearities	70
4.3.5	Another Form of Scalar Time-Delay System	73
4.3.6	El Niño and the Delayed Action Oscillator	76
4.4	Coupled Chaotic Time-Delay Systems	78
4.4.1	Time-Delayed Chua's Circuit	78
4.4.2	Semiconductor Lasers	79
4.4.3	Neural Networks	81
	References	82

5	Implications of Delay Feedback: Amplitude Death and Other Effects	85
5.1	Introduction	85
5.2	Time-Delay Induced Amplitude Death	85
5.2.1	Theoretical Study: Single Oscillator	86
5.2.2	Experimental Study	89
5.3	Amplitude Death with Distributed Delay in Coupled Limit Cycle Oscillators	91
5.4	Amplitude Death in Coupled Chaotic Oscillators	93
5.5	Amplitude Death with Conjugate (Dissimilar) Coupling	96
5.6	Amplitude Death with Dynamic Coupling	98
5.7	Time-Delay Induced Bifurcations	101
5.8	Some Other Effects of Delay Feedback	102
	References	103
6	Recent Developments on Delay Feedback/Coupling: Complex Networks, Chimeras, Globally Clustered Chimeras and Synchronization	105
6.1	Introduction	105
6.2	Complex Networks	105
6.3	Chimera States in Delay Coupled Identical Oscillators	108
6.3.1	Discovery of Chimera States	108
6.3.2	Chimera States in Delay Coupled Systems	111
6.4	Chimera States in Delay Coupled Subpopulations: Globally Clustered States	113
6.5	Synchronization in Complex Networks with Delay	117
6.6	Controlling Using Time-Delay Feedback	118
6.6.1	Pyragas Time-Delay Feedback Control	119
6.6.2	Transient Behavior with Time-Delay Feedback	122
6.7	Further Developments	124
	References	125
7	Complete Synchronization of Chaotic Oscillations in Coupled Time-Delay Systems	127
7.1	Introduction	127
7.2	Complete Synchronization in Coupled Time-Delay Systems	129
7.3	Stability Using Krasovskii-Lyapunov Theory	130
7.4	Numerical Confirmation	133
7.4.1	Case 1	134
7.4.2	Case 2	134
7.4.3	Case 3	135
7.4.4	Case 4	135
7.5	Conclusion	135
	References	136

8	Transition from Anticipatory to Lag Synchronization via Complete Synchronization	139
8.1	Introduction	139
8.2	Coupled System and the General Stability Condition	139
8.3	Coupled Piecewise Linear Time-Delay System and Stability Condition: Transition from Anticipatory to Lag Synchronization	141
8.3.1	Anticipatory Synchronization for $\tau_2 < \tau_1$	142
8.3.2	Complete Synchronization for $\tau_2 = \tau_1$	146
8.3.3	Lag Synchronization for $\tau_2 > \tau_1$	147
8.3.4	Inverse Synchronizations	149
8.4	Transition from Anticipatory to Lag via Complete Synchronization: Mackey-Glass and Ikeda Systems	153
8.4.1	Anticipatory Synchronization for $\tau_2 < \tau_1$	154
8.4.2	Complete Synchronization for $\tau_2 = \tau_1$	158
8.4.3	Lag Synchronization for $\tau_2 > \tau_1$	158
8.5	Inverse Synchronizations: Mackey-Glass and Ikeda Systems	161
	References	163
9	Intermittency Transition to Generalized Synchronization	165
9.1	Introduction	165
9.2	Broad Range (Slow/Delayed) Intermittency Transition to GS for Linear Error Feedback Coupling of the Form $(x_1(t) - x_2(t))$	166
9.3	Stability Condition	167
9.4	Approximate (Intermittent) Generalized Synchronization	168
9.5	Characterization of IGS	171
9.6	Narrow Range (Immediate) Intermittency Transition to GS for Linear Direct Feedback Coupling of the Form $x_1(t)$	173
9.7	Broad Range Intermittency Transition to GS for Nonlinear Error Feedback Coupling of the Form $(f(x_1(t - \tau_2)) - f(x_2(t - \tau_2)))$	178
9.8	Narrow Range Intermittency Transition to GS for Nonlinear Direct Feedback Coupling of the Form $f(x_1(t - \tau_2))$	181
9.9	Intermittency Transition to Generalized Synchronization: Mackey-Glass & Ikeda Systems	185
9.9.1	Broad Range Intermittency Transition to GS	186
9.9.2	Narrow Range Intermittency Transition to GS	190
	References	199
10	Transition from Phase to Generalized Synchronization	201
10.1	Introduction	201
10.2	Phase-Coherent and Non-phase-coherent Attractors	202
10.3	CPS in Chaotic Systems	203
10.4	CPS and Time-Delay Systems	205
10.5	CPS from Poincaré Surface of Section of the Transformed Attractor	207

10.6	CPS from Recurrence Quantification Analysis	210
10.7	CPS from the Lyapunov Exponents	213
10.8	Concept of Localized Sets	214
10.9	Transition from Phase to Generalized Synchronization: Mackey-Glass & Ikeda Systems	215
10.9.1	CPS from Poincaré Section of the Transformed Attractor ..	217
10.9.2	CPS from Recurrence Quantification Analysis	218
10.9.3	CPS from the Lyapunov Exponents	219
10.9.4	CPS in Coupled Ikeda Systems	221
10.10	Summary	225
	References	225
11	DTM Induced Oscillating Synchronization	227
11.1	Introduction	227
11.2	Estimation of the Effect of Delay Time Modulation	228
11.2.1	Filling Factor	228
11.2.2	Length of Polygon Line	230
11.2.3	Average Mutual Information	230
11.3	Coupled System and Stability Condition in the Presence of Delay Time Modulation	233
11.4	Oscillating Synchronization	235
11.5	Intermittent Anticipatory Synchronization	238
11.6	Complete Synchronization	241
11.7	Intermittent Lag Synchronization	242
11.8	Complex Oscillating Synchronization	245
11.9	DTM Induced Oscillating Synchronization: Mackey-Glass & Ikeda Systems	245
11.9.1	Coupled Mackey-Glass Systems	245
11.9.2	Coupled Ikeda Systems	246
11.10	Summary	248
	References	249
12	Exact Solutions of Certain Time Delay Systems: The Car-Following Models	251
12.1	Introduction	251
12.2	The Car-Following Models	251
12.3	The Newell Model	252
12.4	The tanh Car-Following Model	255
12.5	Other Developments	257
	References	258

Appendix

A	Computing Lyapunov Exponents for Time-Delay Systems	259
A.1	Introduction	259
A.2	Lyapunov Exponents of an n -Dimensional Dynamical System	259
A.2.1	Computation of Lyapunov Exponents	260
A.3	Lyapunov Exponents of a DDE	261
	References	262
B	A Brief Introduction to Synchronization in Chaotic Dynamical Systems	263
B.1	Introduction	263
B.2	Characterization of Synchronization	265
B.2.1	Complete Synchronization	268
B.2.2	Phase Synchronization	269
B.2.3	Lag Synchronization	271
B.2.4	Anticipatory Synchronization	272
B.2.5	Generalized Synchronization	273
	References	275
C	Recurrence Analysis	279
C.1	Introduction	279
C.2	Recurrence Plots and Their Variants	280
C.2.1	Recurrence Plots	280
C.2.2	Cross Recurrence Plots (CRP)	282
C.2.3	Joint Recurrence Plots (JRP)	283
C.3	Recurrence Quantification Analysis (RQA)	284
C.3.1	Generalized Autocorrelation Function, $P(t)$	284
C.3.2	Correlation of Probability of Recurrence (CPR)	285
C.3.3	Joint Probability of Recurrence (JPR)	285
C.3.4	Similarity of Probability of Recurrence (SPR)	286
C.4	Synchronization and Recurrences	286
C.4.1	PS in Mutually Coupled Rössler Systems	286
C.4.2	Phase to Lag Synchronization	288
	References	291
D	Some More Examples of DDEs	293
D.1	Introduction	293
D.2	DDEs with Constant Delay	293
D.2.1	Hutchinson's Equation/Delayed Logistic Equation	293
D.2.2	Gopalsamy and Ladas Population Model	293
D.2.3	Stem-Cell Model	294
D.2.4	Pupil Cycling Model	294

D.3	DDEs with Discrete Delays	295
D.3.1	Australian Blowfly Model	295
D.3.2	Wilson and Cowan Model	295
D.3.3	Human Respiratory Model	295
D.4	DDEs with Distributed Delay	296
D.4.1	Volterra's Logistic Equation	296
D.4.2	Neural Network with Distributed Delay	296
D.4.3	Chemostat Model	297
D.5	DDEs with State-Dependent Delay	297
D.5.1	Population Model	297
D.5.2	Logistic Model with State Dependent Delay	297
D.5.3	Mechanical Model for Machine Tool Chatter	298
D.6	DDEs with Time-Dependent Delay	298
D.6.1	Stem-Cell Equation	298
D.6.2	Neural Network Model	298
	References	299
	Glossary	301
	Index	309