

Contents

**Historical foreword on the centenary after Felix Hausdorff's classic
Set Theory — xiii**

Preface — xvii

1 Fundamentals of the theory of classes, sets, and numbers — 1

Introduction — 1

1.1 Classes and sets — 2

**1.1.1 Symbols, symbol-strings, and texts of the theory of classes
and sets — 2**

1.1.2 Formulas and terms — 3

1.1.3 Axioms, deducibility, and theorems — 5

1.1.4 Logical axiom schemes of the theory of classes and sets — 9

**1.1.5 First non-logical axioms and axiom schemes of the theory of classes
and sets — 10**

1.1.6 First axioms of existence of sets — 13

1.1.7 Correspondences — 16

1.1.8 Mappings — 17

1.1.9 Multivalued and simple collections — 22

1.1.10 The union and intersection of a multivalued collection — 25

1.1.11 The other axioms of existence of sets — 27

1.1.12 The product of a multivalued collection. The axiom of choice — 33

**1.1.13 Formulas of the distributivity for union, intersection, and product
of a multivalued collection — 40**

**1.1.14 Binary relations. Equivalence relations. Preorder and order
relations — 43**

1.1.15 Basic notions connected with preorder and order relations — 46

1.2 Ordinals and ordinal numbers — 55

1.2.1 The property of minimality. The principle of induction — 55

1.2.2 The relation of Neumann on the universal class. Ordinals — 56

1.2.3 Properties of ordinals — 58

1.2.4 Relations between well-ordered sets — 61

**1.2.5 The correspondence between well-ordered sets and ordinal
numbers — 62**

1.2.6 Natural numbers. Multivalued and simple sequences — 65

1.2.7 The construction of mappings by natural induction — 69

**1.2.8 The principle of transfinite induction. The constructions of mappings
by transfinite induction — 73**

1.2.9	The ordered disjoint union of well-ordered sets. The addition of ordinal numbers —	75
1.2.10	The connection between ordinal and natural numbers —	77
1.2.11	The other forms of the axiom of choice —	79
1.3	Cardinal numbers —	82
1.3.1	The definition of cardinal numbers. The cardinality of natural numbers. The first denumerable cardinal number —	82
1.3.2	The power of sets —	83
1.3.3	Properties of finite sets —	85
1.3.4	The first uncountable cardinal number. The enumeration of infinite cardinal numbers —	89
1.3.5	Derivative cardinal numbers —	90
1.3.6	Derivative natural numbers —	98
1.3.7	Ordered sets of natural numbers —	104
1.3.8	Properties of infinite cardinal numbers —	106
1.3.9	Properties of countable sets —	110
1.3.10	Properties of the class of all countable ordinal numbers —	111
1.4	Real numbers —	113
1.4.1	Integers —	113
1.4.2	Rational numbers —	126
1.4.3	Real and extended real numbers —	137
1.4.4	The Cantor completeness of the real line —	152
1.4.5	The Dedekind completeness and order properties of the extended real line —	157
1.4.6	Natural roots of positive real numbers. Raising to a rational degree —	164
1.4.7	Convergence of nets in the extended real line —	168
1.4.8	Netful and sequential series in the extended real line —	176
1.4.9	The order equivalence of intervals of the real line —	182
A	Characterization of all natural models of Neumann – Bernays – Gödel and Zermelo – Fraenkel set theories —	185
	Introduction —	185
A.1	First-order theories —	187
A.1.1	The language of first-order theories —	187
A.1.2	Deducibility in a first-order theory —	189
A.1.3	An interpretation of a first-order theory in a set theory —	191
A.2	Some elements of the Zermelo – Fraenkel set theory —	194
A.2.1	The proper axioms and axiom schemes of the ZF set theory —	194
A.2.2	Ordinals and cardinals in the ZF set theory —	198
A.3	Cumulative sets in the ZF set theory —	204
A.3.1	Construction of cumulative sets —	204

A.3.2	Properties of cumulative sets —	205
A.3.3	Properties of inaccessible cumulative sets —	210
A.4	Universal sets and their connection with inaccessible cumulative sets —	213
A.4.1	Universal sets and their properties —	213
A.4.2	Description of the class of all universal sets —	216
A.4.3	Enumeration of the class of all universal sets in the ZF+AU set theory and the structural form of the universality axiom —	218
A.4.4	Enumeration of the class of all inaccessible cardinals in the ZF+AI theory and the structural form of the inaccessibility axiom —	222
A.5	Weak forms of the universality and inaccessibility axioms —	224
A.5.1	The ω -universality and ω -inaccessibility axioms —	224
A.5.2	Comparison of various forms of the universality and inaccessibility axioms —	229
A.6	Characterization of all supertransitive standard models of the ZF and NBG set theories in the ZF set theory —	235
A.6.1	Supertransitive standard model sets with the strong substitution property for the ZF set theory —	235
A.6.2	Supertransitive standard model of the NBG set theory in the ZF set theory —	242
A.7	Characterization of all natural models of the NBG set theory —	251
A.7.1	Tarski sets and their properties —	251
A.7.2	Galactic sets and their connection with Tarski sets —	256
A.7.3	Characterization of Tarski sets. Characterization of all natural models of the NBG theory —	258
A.8	Characterization of all natural models of the ZF set theory in the ZF set theory —	260
A.8.1	Scheme-inaccessible cardinal numbers and scheme-inaccessible cumulative sets —	260
A.8.2	Scheme-universal sets and their connection with scheme-inaccessible cumulative sets —	265
A.8.3	Supertransitive standard models of the ZF set theory in the ZF set theory —	271
A.8.4	Tarski scheme sets. Characterization of all natural models of the ZF set theory —	277
B	Local theory of sets as a foundation for category theory and its connection with the Zermelo – Fraenkel set theory —	281
	Introduction —	281
B.1	The local theory of sets —	285
B.1.1	Proper axioms and axiom schemes of the local theory of sets —	285
B.1.2	Some constructions in the local theory of sets —	290

- B.2 The MacLane problem on a set-theoretical foundation for the naive category theory. The solution of this problem within the framework of the local theory of sets — 291**
 - B.2.1 The definition of a local category in the local theory of sets — 292**
 - B.2.2 Functors and natural transformations and generated by them “the category of categories” and “the category of functors” in the local theory of sets — 293**
- B.3 Universal classes, ordinals, cardinals, and cumulative classes in the local theory of sets — 295**
 - B.3.1 The relativization of formulas of the LTS to universal classes. The interpretation of the ZF set theory in universal classes — 295**
 - B.3.2 The globalization of local constructions — 297**
 - B.3.3 Ordinals and cardinals in the local theory of sets — 300**
 - B.3.4 Cumulative classes in the LTS and their connection with universal classes — 305**
 - B.3.5 The structure of the assemblies of all universal classes and all inaccessible cardinals in the local theory of sets — 308**
- B.4 Relative consistency between the LTS and the ZF set theory — 312**
 - B.4.1 Additional axioms on inaccessible cardinals in the ZF set theory — 312**
 - B.4.2 “Forks” of relative consistency — 317**
- B.5 The proof of relative consistency by the method of abstract interpretation — 330**
 - B.5.1 Abstracts of a set theory — 331**
 - B.5.2 The abstract interpretation of a first-order theory in a set theory — 331**
- B.6 Undeducibility of some axioms in the LTS — 333**
 - B.6.1 The undeducibility of the axiom scheme of replacement — 333**
 - B.6.2 The independence of axiom $AU(\omega)$ of the axioms of the LTS — 344**
 - B.6.3 The locally minimal theory of sets — 345**
- B.7 The finite axiomatizability of the LTS and the NBG set theory — 346**
 - B.7.1 Replacement of the full comprehension axiom scheme by finitely many axioms — 347**
 - B.7.2 The deductive equivalence of the theories LTS and LTS^f — 349**
 - B.7.3 The finite axiomatization of the NBG set theory by P. Bernays — 356**

C Compactness theorem for generalized second-order language — 361

Introduction — 361

C.1 Types, formations, terminals, signatures, and formulas — 363

C.1.1 Types — 363

C.1.2 Formations and terminals — 364

C.1.3 Signatures and formulas — 364

- C.2 Mathematical systems of the signature Σ^g with generalized equalities and belongings — 366**
 - C.2.1 The definition of a mathematical system of the generalized signature Σ^g — 366**
 - C.2.2 Concordance of mathematical systems of the generalized second-order signature — 367**
 - C.2.3 Evaluations and models — 367**
 - C.2.4 The generalized equality of values of evaluations and satisfiability — 369**
 - C.2.5 An example of a good model for the second-order equality axioms — 372**
- C.3 Infraproducts, infrafiltration, and generalized compactness theorem — 373**
 - C.3.1 Infraproducts of collections of evaluated mathematical systems of the generalized second-order signature Σ_2^g — 373**
 - C.3.2 Infrafiltration of formulas of the second-order language $L(\Sigma_2^g)$ of the generalized second-order signature Σ_2^g — 375**
 - C.3.3 Compactness theorem for formulas of the language $L(\Sigma_2^g)$ of the generalized second-order signature — 382**
 - C.3.4 Uncountable models of the second-order generalized Peano – Landau arithmetic — 382**

Index of terms — 387

Index of notations — 409

Bibliography — 417