

Contents

Preface — V

List of Figures — XI

1 Introduction — 1

2 Vectorial Approach — 9

2.1 An Introduction to Vectors and Tensors — 9

2.2 The Four Elementary Tensor Operations — 13

2.3 Rotations in Two Dimensions — 15

2.3.1 Rotation by $\pi/2$ — 16

2.3.2 Tensorial Description of a Rotation — 17

2.3.3 Group Theory Aspects — 18

2.4 Diagrammatic Representation of Tensors — 21

2.5 Rotations in Three Dimensions — 26

2.5.1 The 3-Dimensional Rotation Group — 29

2.5.2 Eulerian Angles — 32

2.5.3 Passive Rotations — 37

3 Tensors and Their Rotational Reduction — 41

3.1 The Rotational Invariance of $\mathbf{U}$ and ε — 45

3.2 Invariant Embedding and Tensorial Reduction — 47

3.2.1 Invariant Embedding — 47

3.2.2 Reduction and Natural Tensors — 48

3.3 The Natural Projection Tensors $\mathbf{E}^{(p)}$ — 51

3.3.1 Contraction of $\mathbf{E}^{(p)}$ — 53

3.3.2 Selection of a direction of $\mathbf{E}^{(p)}$ — 54

3.4 Natural Tensors as Irreducible Representations — 55

3.4.1 Dimension of Symmetric Traceless Tensors of Order p — 56

3.4.2 The Casimir Invariant Eigenvalue for a Symmetric Traceless Tensor — 56

3.5 Parentage Scheme — 57

3.6 Method of Characters — 59

3.6.1 Group Density — 59

3.6.2 Classification of Tensors of Order p via Characters — 61

4 Tensors of $\mathbf{r}$ — 67

4.1 Natural Tensors of $\mathbf{r}$ — 67

4.2 Integrals of Products of $\hat{\mathbf{r}}$ — 70

4.3 Orthogonal Functions of $\mathbf{r}$ Associated with a Gaussian Weight Function — 72

5 Spherical Vectors and Tensors — 75

- 5.1 Group Generator Properties — 76
- 5.2 The Components of a Spherical Vector — 79
- 5.3 Rotation of a Spherical Vector — 82
- 5.4 Spherical Tensors — 84
- 5.5 Spherical Harmonics — 90

6 3- j Coupling Tensors — 93

- 6.1 Definitions — 94
- 6.2 Detailed Calculation of $\mathbf{T}(\ell_1, \ell_2, \ell_3)$ — 96
 - 6.2.1 $\mathbf{T}(\ell_1, \ell_2, \ell_3)$ for L Even — 96
 - 6.2.2 The Special Case that $\ell_1 + \ell_2 = \ell_3$ — 99
 - 6.2.3 The General Case for L Even — 100
 - 6.2.4 $\mathbf{T}(\ell_1, \ell_2, \ell_3)$ for L Odd — 102
- 6.3 Detailed Calculation of $\Omega(\ell_1, \ell_2, \ell_3)$ — 104
 - 6.3.1 $\Omega(\ell_1, \ell_2, \ell_3)$ for L Even — 105
 - 6.3.2 $\Omega(\ell_1, \ell_2, \ell_3)$ for L Odd — 107
- 6.4 Three Basic Contractions — 108
- 6.5 Recursion Relations — 111
- 6.6 Calculation of $\sum_{abc} x_{a,b,c}$ — 115
 - 6.6.1 Sum over a and b — 116
 - 6.6.2 Sum over c — 117
 - 6.6.3 Calculation of $\sum_{ab} x_{a,b,0}$ — 117
 - 6.6.4 $\sum_{abc} x_{a,b,c}$ for Even and Odd L — 121

7 Properties of 3- j Tensors — 123

- 7.1 Completeness Relation — 123
- 7.2 Integrals over Functions of $\hat{\mathbf{r}}$ — 124
- 7.3 3- j Symbols — 128
 - 7.3.1 Basic Properties of 3- j Symbols — 129
- 7.4 Integrals of Spherical Harmonics — 131
- 7.5 Invariant Functions of Three Vectors — 132
- 7.6 Calculation of Two Particular Sets of 3- j Symbols — 137

8 The 6- j and Other n - j Symbols — 139

- 8.1 The 6- j Symbol — 139
- 8.2 6- j Symbol Properties — 142
- 8.3 Evaluation of Certain 6- j Symbols — 144
 - 8.3.1 The Cases when $\ell = 0$ and 1 — 144

8.3.2	The Case when $\ell_3 = \ell_1 + \ell_2$ —	147
8.3.3	A Recursion Relation —	151
8.4	The $9-j$ Symbol —	152
9	Rotation Matrices —	155
9.1	Axis-Angle Representation —	155
9.2	Euler Angle Representation —	158
9.2.1	Alternate Formula —	159
9.3	Connection with Spherical Harmonics —	160
9.4	Properties of the Rotation Matrices —	163
9.5	As Eigenvectors of the Rotation Generators —	167
10	Spinors —	169
10.1	The Standard Spinor Basis Sets —	170
10.2	Rotation of a Spinor —	172
10.2.1	Products of Rotations —	175
10.3	The Invariant Tensor ϵ —	176
10.4	Spinor Invariants —	180
10.5	Reduction of Spinor Tensors of Order p —	182
10.6	Spinor Tensors of Weight ℓ —	185
10.7	$3-j$ Spinor Tensors —	188
10.7.1	Normalization Calculation —	189
10.7.2	$3-j$ Symbols —	191
10.8	Spinor-Cartesian Transformation —	193
10.8.1	Cayley-Klein Representation of a Vector —	196
10.9	Discussion of Spinors —	197
11	Applications to Quantum Mechanics —	199
11.1	Eigenvectors of the Angular Momentum —	200
11.1.1	Abstract Properties —	201
11.1.2	Position Representation —	203
11.1.3	Tensorial Representation —	207
11.1.4	Momentum Representation —	207
11.2	Symmetric Top Eigenvectors —	209
11.2.1	Anomalous Commutation Relations —	213
11.3	Wigner-Eckart Theorem —	214
11.4	Tensors of Angular Momentum —	217
11.5	Traces of Products of J —	221
11.6	Angular Momentum Superoperators —	225
A	Some Formalities of Linear Algebra —	229
A.1	Group Theory —	229

X — Contents

A.2 Vector Spaces — 234

A.3 Cartesian Tensors and Polyadics — 242

B Calculation of $D_{mn}^{(\ell)}(0, \beta, 0)$ — 247

Bibliography — 251

Symbol Index — 253