

Contents

1	Basic Notions	1
1.1	Structures	1
1.1.1	Finite Words and ω -words	2
1.1.2	Infinite Trees	3
1.2	Logic	5
1.3	Games	6
1.3.1	Positional Strategies	6
1.3.2	Parity Games	7
1.4	Automata	7
1.4.1	Parity Index of an Automaton	10
1.5	Algebra	10
1.5.1	Semigroups and Monoids	11
1.5.2	Wilke Algebras	11
1.5.3	Recognition	13
1.5.4	Ramsey's Theorem for Semigroups	13
1.6	Topology	14
1.6.1	Borel and Projective Hierarchy	14
1.6.2	Topological Complexity	16
1.6.3	Ranks	17
1.6.4	The Boundedness Theorem	17
1.6.5	Co-inductive Definitions	18
1.7	Regular Languages	19
1.7.1	Classes of Regular Tree Languages	21
1.7.2	Index Hierarchies	22
1.7.3	Topological Complexity of Regular Languages	23
1.7.4	The languages $W_{i,j}$	23
1.7.5	Separation Property	24
1.7.6	Deterministic Languages	26

Subclasses of Regular Languages

2	Introduction	29
2.1	Motivations	30
2.1.1	Definability in wmsO	30
2.1.2	Index Problem	30
2.1.3	Bi-unambiguous Languages	31
2.1.4	Borel Languages	32
2.1.5	Topological Complexity vs. Decidability	32
2.1.6	Separation Property	33
2.2	Overview of the Parts	33
2.2.1	Part I: Subclasses of Regular Languages	34
2.2.2	Part II: Thin Algebras	35
2.2.3	Part III: Extensions of Regular Languages	36
3	Collapse for Unambiguous Automata	37
3.1	Unique Runs	38
3.2	Construction of the Automaton	40
3.2.1	Soundness	41
3.2.2	Completeness	42
3.3	Conclusions	43
4	When a Büchi Language Is Definable in wmsO	45
4.1	The Ordinal of a Büchi Automaton	46
4.1.1	Truncated Runs	47
4.1.2	The Reduction	48
4.1.3	Ranks	50
4.2	Extending Runs	52
4.2.1	K -reach Implies Big Rank	53
4.2.2	Big Rank Implies K -reach	54
4.3	Automata for K -safe Trees	56
4.4	Boundedness Game	58
4.4.1	Rules of the Game	59
4.4.2	Winning Condition	61
4.5	Equivalence	62
4.5.1	Implication $(1) \Rightarrow (2)$	62
4.5.2	Implication $(2) \Rightarrow (1)$	66
4.6	Conclusions	68
5	Index Problems for Game Automata	71
5.1	Runs of Game Automata	72
5.2	Non-deterministic Index Problem	74
5.3	Partial Objects	76
5.3.1	Trees	76
5.3.2	Games	77
5.3.3	Automata	77

5.3.4 Composing Automata	78
5.3.5 Resolving	78
5.4 Alternating Index Problem	79
5.4.1 The Algorithm	79
5.4.2 Upper Bounds	81
5.4.3 Lower Bounds	84
5.5 Conclusions	89

Thin Algebras

6 When a Thin Language Is Definable in wms0	93
6.1 Basic Notions	95
6.1.1 Thin Trees	95
6.1.2 Automata	96
6.1.3 Examples of Skurczyński	97
6.1.4 Ranks	98
6.1.5 Skeletons	100
6.2 Thin Algebra	102
6.2.1 The Automaton Algebra	104
6.3 Upper Bounds	105
6.3.1 Embeddings and Quasi-skeletons	106
6.3.2 Proof of Proposition 6.13	109
6.4 Characterisation of wms0-definable Languages	110
6.4.1 Implication (2) $\Rightarrow$ (3)	110
6.4.2 Implication (5) $\Rightarrow$ (1)	115
6.5 Conclusions	118
7 Recognition by Thin Algebras	121
7.1 Prophetic Thin Algebras	122
7.2 Bi-unambiguous Languages	125
7.2.1 Prophetic Thin Algebras Recognise only Bi-unambiguous Languages	125
7.2.2 Markings by the Automaton Algebra for an Unambiguous Automaton	126
7.2.3 Every Bi-unambiguous Language is Recognised by a Prophetic Thin Algebra	129
7.3 Consequences of Conjecture 2.1	130
7.4 Decidable Characterisation of the Bi-unambiguous Languages	131
7.4.1 Pseudo-syntactic Morphisms	132
7.4.2 Decidable Characterisation	134
7.5 Conclusions	135
8 Uniformization on Thin Trees	137
8.1 Basic Notions	138
8.2 Transducer for a Uniformized Relation	139

8.3	Choice Hypothesis	143
8.3.1	Equivalence $(1) \Leftrightarrow (2)$	144
8.3.2	Implication $(2) \Rightarrow (3)$	145
8.3.3	Implication $(4) \Rightarrow (2)$	150
8.4	Negative Results	151
8.4.1	Green's Relations	151
8.4.2	A Marking of a Thick Tree	152
8.4.3	Non-uniformizability of Skeletons	154
8.4.4	Ambiguity of Thin Trees	155
8.5	Conclusions	156

Extensions of Regular Languages

9	Descriptive Complexity of $\text{MSO}+\text{U}$	159
9.1	Basic Notions	160
9.1.1	Languages IF^i	162
9.2	Languages H_i	163
9.3	Functions c_i , d_i , and r_i	166
9.4	Reductions	168
9.5	Upper Bounds	169
9.5.1	Proof of Theorem 9.7	170
9.6	Conclusions	171
10	Undecidability of $\text{MSO}+\text{U}$	173
10.1	Basic Notions	175
10.1.1	Gödel's Constructible Universe	175
10.1.2	Projective MSO	176
10.2	Reduction	177
10.3	Projective Determinacy	178
10.4	Undecidability of proj-MSO on $\{L, R\}^\omega$	180
10.4.1	Proof of Theorem 10.88	180
10.5	Conclusions	181
11	Separation for $\omega\text{B-}$ and $\omega\text{S-regular Languages}$	183
11.1	Basic Notions	184
11.1.1	Monoid of Runs	184
11.1.2	Profinite Monoid	185
11.1.3	Ramsey-Type Arguments	187
11.1.4	Notation	188
11.2	Automata	188
11.2.1	$\omega\text{B-}$ and $\omega\text{S-automata}$	189
11.2.2	B- and S-automata	190
11.2.3	Languages	191
11.3	Separation for Profinite Languages	193

	Contents	XIII
11.4	Reduction	195
11.4.1	Implication $(1) \Rightarrow (2)$.	197
11.4.2	Implication $(2) \Rightarrow (3)$.	199
11.4.3	Implication $(3) \Rightarrow (1)$.	199
11.5	Separation for ω -languages	200
11.6	Conclusions	202
12	Conclusions	205
	References.	207