

Contents

Introduction	1 [1]
The Aim of this Lecture Course	1 [1]
Relevant Literature	3 [4]
 First Part: Arithmetic	 7 [6]
I. Calculating with Natural Numbers	9 [6]
1. Introduction of Numbers in the Schools	9 [6]
2. The Fundamental Laws of Reckoning	11 [9]
3. The Logical Foundations of Operations with Integers	13 [11]
4. Practice in Calculating with Integers	19 [18]
Description of the Calculating Machine “Brunsviga”	19 [19]
 II. The First Extension of the Notion of Number	25 [24]
1. Negative Numbers	25 [24]
On the History of Negative Numbers	27 [27]
2. Fractions	31 [31]
3. Irrational Numbers	33 [34]
On the Nature of Space Intuition (Precision and Approximation Mathematics)	37 [38]
 III. Concerning Special Properties of Integers	39 [40]
Number Theory in Schools and in Universities	39 [40]
Special Discussions of Number-theoretic Issues	41 [43]
Prime Numbers, Decomposition into Prime Factors	41 [43]
Transformation of Ordinary Fractions into Decimal Numbers	42 [44]
Continued Fractions	43 [45]
Pythagorean Numbers, Fermat’s Last Theorem	47 [49]
The Problem of the Division of the Circle	52 [54]

IV. Complex Numbers	59 [61]
1. Ordinary Complex Numbers	59 [61]
2. Higher Complex Numbers, Especially Quaternions	61 [64]
Remarks on Vector Calculus	67 [69]
3. Quaternion Multiplication – Rotation-Dilation	69 [71]
Interpretation in Three-dimensional Space	72 [74]
4. Complex Numbers in School Teaching	79 [81]
Concerning the Modern Development and the General Structure of Mathematics	81 [82]
The Structure of Elementary Analysis According to Two Parallel Processes of Development of Differing Character	81 [82]
Overview of the History of Mathematics	84 [86]
Second Part: Algebra	89 [93]
Textbooks	89 [93]
Our Specific Goal: Application of Geometrically Intuitive Methods to the Solution of Equations	90 [93]
I. Real Equations with Real Unknowns	91 [94]
1. Equations with One Parameter	91 [94]
2. Equations with Two Parameters	92 [95]
Classification According to the Number of Real Roots	97 [99]
3. Equations with Three Parameters λ, μ, ν	100 [101]
An Apparatus for the Numerical Solution of Equations	101 [102]
The Discriminant Surface of the Biquadratic Equation	102 [103]
II. Equations in the Field of Complex Quantities	109 [109]
A. The Fundamental Theorem of Algebra	109 [109]
B. Equations with a Complex Parameter	112 [112]
1. The “Pure” Equation	118 [119]
Irreducibility; “Impossibility” of Trisecting the Angle	121 [122]
2. The Dihedral Equation	124 [124]
3. The Tetrahedral, the Octahedral, and the Icosahedral Equations	129 [130]
4. Continuation: Setting up the Normal Equation	133 [134]
5. Concerning the Solution of the Normal Equations	139 [140]
6. Uniformisation of the Normal Irrationalities by Means of Transcendental Functions	142 [143]
The “Casus Irreducibilis” and the Trigonometric Solution of the Cubic Equation	144 [145]
7. Solvability in Terms of Radical Signs	147 [148]
8. Reduction of General Equations to Normal Equations	150 [152]

Third Part: Analysis 153 [155]

I. Logarithmic and Exponential Functions 155 [155]

1. Systematic Account of Algebraic Analysis 155 [155]

2. The Historical Development of the Theory 157 [157]

 Neper and Bürgi 158 [158]

 The 17th Century: The Area of the Hyperbola 162 [160]

 Euler and Lagrange: Algebraic Analysis 163 [164]

 The 19th Century: Functions of Complex Variables 165 [166]

3. Remarks about Teaching in Schools 166 [167]

4. The Standpoint of Function Theory 168 [169]

 The Passage to the Limit from the Power
 to the Exponential Function 172 [173]

II. The Goniometric Functions 175 [175]

1. Theory of the Goniometric Functions 175 [175]

 Exact Comparison with the Theory of Logarithms 177 [176]

2. Trigonometric Tables 184 [183]

 A. Purely Trigonometric Tables 184 [183]

 B. Logarithmic-Trigonometric Tables 186 [185]

3. Applications of Goniometric Functions 189 [188]

 A. Trigonometry, in Particular, Spherical Trigonometry . . . 189 [189]

 Formulas of Second Kind: Triangles of First and
 Second Kind 194 [194]

 The Area of Spherical Triangles 197 [197]

 B. Theory of Small Oscillations, Especially Those
 of the Pendulum 201 [201]

 Teaching in Schools (Hidden Infinitesimal Calculus) . . 202 [202]

 C. Representation of Periodic Functions by Means of Series
 of Goniometric Functions (Trigonometric Series) . . . 205 [205]

 Approximation by Series with a Finite Number
 of Terms 206 [206]

 Estimation of Errors; Convergence of Infinite Series . . 211 [211]

 The Gibbs Phenomenon 214 [214]

 Excursions Concerning the General Notion
 of Function 216 [215]

 Historical Importance of Trigonometric Series;
 the Role of Fourier 221 [221]

III. Concerning Infinitesimal Calculus Proper 225 [223]

1. General Considerations in Infinitesimal Calculus 225 [223]

 Emergence of the Infinitesimal Calculus by the Specificity
 of Our Sense Intuition 226 [224]

 The Logical Foundation of Differential and Integral Calculus
 by Means of the Limit Concept
 (Newton and His Successors up to Cauchy) 229 [228]

Construction of the Infinitesimal Calculus Based
on “Differentials” (Leibniz and His Followers) 232 [231]
The Actual Infinitely Small Quantities in the Axiomatics
of Geometry 236 [234]
The Reaction: the Derivative Calculus of Lagrange 238 [237]
Form and Importance of the Infinitesimal Calculus
in the Present State of Teaching 239 [239]
2. Taylor’s Theorem 241 [241]
The First Parabolas of Osculation 241 [241]
Increasing the Order: Questions of Convergence 245 [244]
Generalising Taylor’s Theorem to a Theorem
of Finite Differences 248 [246]
Cauchy’s Estimate of the Error 251 [249]
Historical Remarks About Taylor and Maclaurin 253 [251]
3. Historical and Pedagogical Considerations 254 [253]
Remarks About Textbooks for the Infinitesimal Calculus . . 254 [253]
Characterising Our Proper Presentation 256 [254]

IV. Supplement 259 [256]
IVa. Transcendence of the Numbers e and π 259 [256]
Historical Aspects 259 [256]
Proof of the Transcendence of e , 260 [257]
Proof of the Transcendence of π 266 [263]
More on Transcendent and Algebraic Numbers 272 [269]
IVb. Set Theory 274 [271]
1. The Potency of Sets 274 [271]
Denumerability of Rational and Algebraic Numbers . 275 [273]
The Continuum Not Being Denumerable 278 [276]
Potency of Continua of More Dimensions 280 [278]
Sets with Higher Potencies 283 [281]
2. Arrangement of the Elements of a Set 286 [283]
Types of Arrangement of Denumerable Sets 286 [284]
The Continuity of the Continuum 287 [285]
Invariance of Dimension for Continuous
One-to-One Representations 288 [286]
Closing Remarks 290 [287]
Importance and Goals of Set Theory 290 [289]

Appendix 1: On the Efforts to Reform Mathematics Teaching 293 [290]
Appendix 2: About Mathematical and Pedagogical Literature . . . 299 [298]
Name Index 305 [304]
Subject Index 309 [306]