

Contents

Preface	ix
1 Basic Notions of Phase Space Analysis	1
1.1 Introduction to pseudo-differential operators	1
1.1.1 Prolegomena	1
1.1.2 Quantization formulas	9
1.1.3 The $S_{1,0}^m$ class of symbols	11
1.1.4 The semi-classical calculus	22
1.1.5 Other classes of symbols	27
1.2 Pseudo-differential operators on an open subset of $\mathbb{R}^n$	28
1.2.1 Introduction	28
1.2.2 Inversion of (micro)elliptic operators	32
1.2.3 Propagation of singularities	37
1.2.4 Local solvability	42
1.3 Pseudo-differential operators in harmonic analysis	51
1.3.1 Singular integrals, examples	51
1.3.2 Remarks on the Calderón-Zygmund theory and classical pseudo-differential operators	54
2 Metrics on the Phase Space	57
2.1 The structure of the phase space	57
2.1.1 Symplectic algebra	57
2.1.2 The Wigner function	58
2.1.3 Quantization formulas	58
2.1.4 The metaplectic group	60
2.1.5 Composition formula	62
2.2 Admissible metrics	67
2.2.1 A short review of examples of pseudo-differential calculi . .	67
2.2.2 Slowly varying metrics on $\mathbb{R}^{2n}$	68
2.2.3 The uncertainty principle for metrics	72
2.2.4 Temperate metrics	74
2.2.5 Admissible metric and weights	76

2.2.6	The main distance function	80
2.3	General principles of pseudo-differential calculus	83
2.3.1	Confinement estimates	84
2.3.2	Biconfinement estimates	84
2.3.3	Symbolic calculus	91
2.3.4	Additional remarks	94
2.3.5	Changing the quantization	100
2.4	The Wick calculus of pseudo-differential operators	100
2.4.1	Wick quantization	100
2.4.2	Fock-Bargmann spaces	104
2.4.3	On the composition formula for the Wick quantization	106
2.5	Basic estimates for pseudo-differential operators	110
2.5.1	L^2 estimates	110
2.5.2	The Gårding inequality with gain of one derivative	113
2.5.3	The Fefferman-Phong inequality	115
2.5.4	Analytic functional calculus	134
2.6	Sobolev spaces attached to a pseudo-differential calculus	137
2.6.1	Introduction	137
2.6.2	Definition of the Sobolev spaces	138
2.6.3	Characterization of pseudo-differential operators	140
2.6.4	One-parameter group of elliptic operators	146
2.6.5	An additional hypothesis for the Wiener lemma: the geodesic temperance	152
3	Estimates for Non-Selfadjoint Operators	161
3.1	Introduction	161
3.1.1	Examples	161
3.1.2	First-bracket analysis	171
3.1.3	Heuristics on condition (Ψ)	174
3.2	The geometry of condition (Ψ)	177
3.2.1	Definitions and examples	177
3.2.2	Condition (P)	179
3.2.3	Condition (Ψ) for semi-classical families of functions	181
3.2.4	Some lemmas on C^3 functions	190
3.2.5	Inequalities for symbols	194
3.2.6	Quasi-convexity	200
3.3	The necessity of condition (Ψ)	203
3.4	Estimates with loss of $k/k + 1$ derivative	205
3.4.1	Introduction	205
3.4.2	The main result on subellipticity	207
3.4.3	Simplifications under a more stringent condition on the symbol	207
3.5	Estimates with loss of one derivative	209
3.5.1	Local solvability under condition (P)	209

3.5.2	The two-dimensional case, the oblique derivative problem . . .	216
3.5.3	Transversal sign changes	220
3.5.4	Semi-global solvability under condition (P)	225
3.6	(Ψ) does not imply solvability with loss of one derivative	226
3.6.1	Introduction	226
3.6.2	Construction of a counterexample	232
3.6.3	More on the structure of the counterexample	246
3.7	Condition (Ψ) does imply solvability with loss of $3/2$ derivatives	250
3.7.1	Introduction	250
3.7.2	Energy estimates	251
3.7.3	From semi-classical to local estimates	263
3.8	Concluding remarks	283
3.8.1	A short historical account of solvability questions	283
3.8.2	Open problems	284
3.8.3	Pseudo-spectrum and solvability	285
4	Appendix	287
4.1	Some elements of Fourier analysis	287
4.1.1	Basics	287
4.1.2	The logarithm of a non-singular symmetric matrix	289
4.1.3	Fourier transform of Gaussian functions	291
4.1.4	Some standard examples of Fourier transform	295
4.1.5	The Hardy Operator	299
4.2	Some remarks on algebra	300
4.2.1	On simultaneous diagonalization of quadratic forms	300
4.2.2	Some remarks on commutative algebra	301
4.3	Lemmas of classical analysis	303
4.3.1	On the Faà di Bruno formula	303
4.3.2	On Leibniz formulas	305
4.3.3	On Sobolev norms	306
4.3.4	On partitions of unity	308
4.3.5	On non-negative functions	310
4.3.6	From discrete sums to finite sums	317
4.3.7	On families of rapidly decreasing functions	319
4.3.8	Abstract lemma for the propagation of singularities	322
4.4	On the symplectic and metaplectic groups	324
4.4.1	The symplectic structure of the phase space	324
4.4.2	The metaplectic group	334
4.4.3	A remark on the Feynman quantization	337
4.4.4	Positive quadratic forms in a symplectic vector space	338
4.5	Symplectic geometry	344
4.5.1	Symplectic manifolds	344
4.5.2	Normal forms of functions	345
4.6	Composing a large number of symbols	346

4.7	A few elements of operator theory	356
4.7.1	A selfadjoint operator	356
4.7.2	Cotlar's lemma	357
4.7.3	Semi-classical Fourier integral operators	361
4.8	On the Sjöstrand algebra	366
4.9	More on symbolic calculus	367
4.9.1	Properties of some metrics	367
4.9.2	Proof of Lemma 3.2.12 on the proper class	368
4.9.3	More elements of Wick calculus	370
4.9.4	Some lemmas on symbolic calculus	374
4.9.5	The Beals-Fefferman reduction	376
4.9.6	On tensor products of homogeneous functions	378
4.9.7	On the composition of some symbols	379
	Bibliography	383
	Index	395