

Contents

1	Some General Mathematical Concepts and Notation	1
1.1	Logical Symbolism	1
1.1.1	Connectives and Brackets	1
1.1.2	Remarks on Proofs	2
1.1.3	Some Special Notation	3
1.1.4	Concluding Remarks	3
1.1.5	Exercises	4
1.2	Sets and Elementary Operations on Them	5
1.2.1	The Concept of a Set	5
1.2.2	The Inclusion Relation	7
1.2.3	Elementary Operations on Sets	8
1.2.4	Exercises	11
1.3	Functions	12
1.3.1	The Concept of a Function (Mapping)	12
1.3.2	Elementary Classification of Mappings	16
1.3.3	Composition of Functions and Mutually Inverse Mappings	17
1.3.4	Functions as Relations. The Graph of a Function	19
1.3.5	Exercises	22
1.4	Supplementary Material	25
1.4.1	The Cardinality of a Set (Cardinal Numbers)	25
1.4.2	Axioms for Set Theory	27
1.4.3	Remarks on the Structure of Mathematical Propositions and Their Expression in the Language of Set Theory	29
1.4.4	Exercises	32
2	The Real Numbers	35
2.1	The Axiom System and Some General Properties of the Set of Real Numbers	35
2.1.1	Definition of the Set of Real Numbers	35
2.1.2	Some General Algebraic Properties of Real Numbers	39

2.1.3	The Completeness Axiom and the Existence of a Least Upper (or Greatest Lower) Bound of a Set of Numbers . .	44
2.2	The Most Important Classes of Real Numbers and Computational Aspects of Operations with Real Numbers . .	46
2.2.1	The Natural Numbers and the Principle of Mathematical Induction	46
2.2.2	Rational and Irrational Numbers	49
2.2.3	The Principle of Archimedes	52
2.2.4	The Geometric Interpretation of the Set of Real Numbers and Computational Aspects of Operations with Real Numbers	54
2.2.5	Problems and Exercises	66
2.3	Basic Lemmas Connected with the Completeness of the Real Numbers	70
2.3.1	The Nested Interval Lemma (Cauchy–Cantor Principle) .	71
2.3.2	The Finite Covering Lemma (Borel–Lebesgue Principle, or Heine–Borel Theorem)	71
2.3.3	The Limit Point Lemma (Bolzano–Weierstrass Principle)	72
2.3.4	Problems and Exercises	73
2.4	Countable and Uncountable Sets	74
2.4.1	Countable Sets	74
2.4.2	The Cardinality of the Continuum	76
2.4.3	Problems and Exercises	77
3	Limits	79
3.1	The Limit of a Sequence	79
3.1.1	Definitions and Examples	79
3.1.2	Properties of the Limit of a Sequence	81
3.1.3	Questions Involving the Existence of the Limit of a Sequence	85
3.1.4	Elementary Facts About Series	95
3.1.5	Problems and Exercises	103
3.2	The Limit of a Function	106
3.2.1	Definitions and Examples	106
3.2.2	Properties of the Limit of a Function	110
3.2.3	The General Definition of the Limit of a Function (Limit over a Base)	126
3.2.4	Existence of the Limit of a Function	130
3.2.5	Problems and Exercises	146
4	Continuous Functions	149
4.1	Basic Definitions and Examples	149
4.1.1	Continuity of a Function at a Point	149
4.1.2	Points of Discontinuity	154
4.2	Properties of Continuous Functions	157
4.2.1	Local Properties	157

4.2.2	Global Properties of Continuous Functions	158
4.2.3	Problems and Exercises	167
5	Differential Calculus	171
5.1	Differentiable Functions	171
5.1.1	Statement of the Problem and Introductory Considerations	171
5.1.2	Functions Differentiable at a Point	175
5.1.3	The Tangent Line; Geometric Meaning of the Derivative and Differential	178
5.1.4	The Role of the Coordinate System	181
5.1.5	Some Examples	183
5.1.6	Problems and Exercises	188
5.2	The Basic Rules of Differentiation	190
5.2.1	Differentiation and the Arithmetic Operations	190
5.2.2	Differentiation of a Composite Function (Chain Rule) . .	194
5.2.3	Differentiation of an Inverse Function	197
5.2.4	Table of Derivatives of the Basic Elementary Functions .	202
5.2.5	Differentiation of a Very Simple Implicit Function	202
5.2.6	Higher-Order Derivatives	206
5.2.7	Problems and Exercises	210
5.3	The Basic Theorems of Differential Calculus	211
5.3.1	Fermat's Lemma and Rolle's Theorem	211
5.3.2	The Theorems of Lagrange and Cauchy on Finite Increments	213
5.3.3	Taylor's Formula	217
5.3.4	Problems and Exercises	230
5.4	The Study of Functions Using the Methods of Differential Calculus	234
5.4.1	Conditions for a Function to be Monotonic	234
5.4.2	Conditions for an Interior Extremum of a Function	235
5.4.3	Conditions for a Function to be Convex	241
5.4.4	L'Hôpital's Rule	248
5.4.5	Constructing the Graph of a Function	250
5.4.6	Problems and Exercises	259
5.5	Complex Numbers and the Connections Among the Elementary Functions	263
5.5.1	Complex Numbers	263
5.5.2	Convergence in $\mathbb{C}$ and Series with Complex Terms	267
5.5.3	Euler's Formula and the Connections Among the Elementary Functions	271
5.5.4	Power Series Representation of a Function. Analyticity .	275
5.5.5	Algebraic Closedness of the Field $\mathbb{C}$ of Complex Numbers	280
5.5.6	Problems and Exercises	285
5.6	Some Examples of the Application of Differential Calculus in Problems of Natural Science	287
5.6.1	Motion of a Body of Variable Mass	287
5.6.2	The Barometric Formula	289

5.6.3	Radioactive Decay, Chain Reactions, and Nuclear Reactors	291
5.6.4	Falling Bodies in the Atmosphere	293
5.6.5	The Number e and the Function $\exp x$ Revisited	295
5.6.6	Oscillations	298
5.6.7	Problems and Exercises	302
5.7	Primitives	306
5.7.1	The Primitive and the Indefinite Integral	306
5.7.2	The Basic General Methods of Finding a Primitive	308
5.7.3	Primitives of Rational Functions	314
5.7.4	Primitives of the Form $\int R(\cos x, \sin x) dx$	318
5.7.5	Primitives of the Form $\int R(x, y(x)) dx$	321
5.7.6	Problems and Exercises	324
6	Integration	331
6.1	Definition of the Integral and Description of the Set of Integrable Functions	331
6.1.1	The Problem and Introductory Considerations	331
6.1.2	Definition of the Riemann Integral	333
6.1.3	The Set of Integrable Functions	335
6.1.4	Problems and Exercises	347
6.2	Linearity, Additivity and Monotonicity of the Integral	349
6.2.1	The Integral as a Linear Function on the Space $\mathcal{R}[a, b]$	349
6.2.2	The Integral as an Additive Function of the Interval of Integration	350
6.2.3	Estimation of the Integral, Monotonicity of the Integral, and the Mean-Value Theorem	352
6.2.4	Problems and Exercises	359
6.3	The Integral and the Derivative	360
6.3.1	The Integral and the Primitive	360
6.3.2	The Newton–Leibniz Formula	363
6.3.3	Integration by Parts in the Definite Integral and Taylor’s Formula	364
6.3.4	Change of Variable in an Integral	366
6.3.5	Some Examples	368
6.3.6	Problems and Exercises	372
6.4	Some Applications of Integration	375
6.4.1	Additive Interval Functions and the Integral	375
6.4.2	Arc Length	377
6.4.3	The Area of a Curvilinear Trapezoid	383
6.4.4	Volume of a Solid of Revolution	384
6.4.5	Work and Energy	385
6.4.6	Problems and Exercises	391
6.5	Improper Integrals	393
6.5.1	Definition, Examples, and Basic Properties of Improper Integrals	393

6.5.2	Convergence of an Improper Integral	398
6.5.3	Improper Integrals with More than One Singularity . . .	404
6.5.4	Problems and Exercises	406
7	Functions of Several Variables: Their Limits and Continuity	409
7.1	The Space $\mathbb{R}^m$ and the Most Important Classes of Its Subsets . .	409
7.1.1	The Set $\mathbb{R}^m$ and the Distance in It	409
7.1.2	Open and Closed Sets in $\mathbb{R}^m$	411
7.1.3	Compact Sets in $\mathbb{R}^m$	413
7.1.4	Problems and Exercises	415
7.2	Limits and Continuity of Functions of Several Variables	416
7.2.1	The Limit of a Function	416
7.2.2	Continuity of a Function of Several Variables and Properties of Continuous Functions	421
7.2.3	Problems and Exercises	426
8	The Differential Calculus of Functions of Several Variables	427
8.1	The Linear Structure on $\mathbb{R}^m$	427
8.1.1	$\mathbb{R}^m$ as a Vector Space	427
8.1.2	Linear Transformations $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$	428
8.1.3	The Norm in $\mathbb{R}^m$	429
8.1.4	The Euclidean Structure on $\mathbb{R}^m$	431
8.2	The Differential of a Function of Several Variables	432
8.2.1	Differentiability and the Differential of a Function at a Point	432
8.2.2	The Differential and Partial Derivatives of a Real-Valued Function	433
8.2.3	Coordinate Representation of the Differential of a Mapping. The Jacobi Matrix	436
8.2.4	Continuity, Partial Derivatives, and Differentiability of a Function at a Point	437
8.3	The Basic Laws of Differentiation	438
8.3.1	Linearity of the Operation of Differentiation	438
8.3.2	Differentiation of a Composition of Mappings (Chain Rule)	441
8.3.3	Differentiation of an Inverse Mapping	446
8.3.4	Problems and Exercises	448
8.4	The Basic Facts of Differential Calculus of Real-Valued Functions of Several Variables	454
8.4.1	The Mean-Value Theorem	454
8.4.2	A Sufficient Condition for Differentiability of a Function of Several Variables	456
8.4.3	Higher-Order Partial Derivatives	457
8.4.4	Taylor's Formula	461
8.4.5	Extrema of Functions of Several Variables	462
8.4.6	Some Geometric Images Connected with Functions of Several Variables	469

8.4.7	Problems and Exercises	474
8.5	The Implicit Function Theorem	480
8.5.1	Statement of the Problem and Preliminary Considerations	480
8.5.2	An Elementary Version of the Implicit Function Theorem	482
8.5.3	Transition to the Case of a Relation $F(x^1, \dots, x^m, y) = 0$	486
8.5.4	The Implicit Function Theorem	489
8.5.5	Problems and Exercises	494
8.6	Some Corollaries of the Implicit Function Theorem	498
8.6.1	The Inverse Function Theorem	498
8.6.2	Local Reduction of a Smooth Mapping to Canonical Form	503
8.6.3	Functional Dependence	508
8.6.4	Local Resolution of a Diffeomorphism into a Composition of Elementary Ones	509
8.6.5	Morse's Lemma	512
8.6.6	Problems and Exercises	515
8.7	Surfaces in $\mathbb{R}^n$ and the Theory of Extrema with Constraint	517
8.7.1	k -Dimensional Surfaces in $\mathbb{R}^n$	517
8.7.2	The Tangent Space	521
8.7.3	Extrema with Constraint	526
8.7.4	Problems and Exercises	539
	Some Problems from the Midterm Examinations	545
1	Introduction to Analysis (Numbers, Functions, Limits)	545
2	One-Variable Differential Calculus	546
3	Integration and Introduction to Several Variables	549
4	Differential Calculus of Several Variables	550
	Examination Topics	555
1	First Semester	555
1.1	Introduction to Analysis and One-Variable Differential Calculus	555
2	Second Semester	557
2.1	Integration. Multivariable Differential Calculus	557
	Appendix A Mathematical Analysis (Introductory Lecture)	559
A.1	Two Words About Mathematics	559
A.2	Number, Function, Law	560
A.3	Mathematical Model of a Phenomenon (Differential Equations, or We Learn How to Write)	561
A.4	Velocity, Derivative, Differentiation	563
A.5	Higher Derivatives, What for?	565
A.5.1	Again Toward Numbers	566
A.5.2	And What to Do Next?	567

Appendix B Numerical Methods for Solving Equations	
(An Introduction)	569
B.1 Roots of Equations and Fixed Points of Mappings	569
B.2 Contraction Mappings and Iterative Process	569
B.3 The Method of Tangents (Newton's Method)	570
Appendix C The Legendre Transform (First Discussion)	573
C.1 Initial Definition of the Legendre Transform and the General Young Inequality	573
C.2 Specification of the Definition in the Case of Convex Functions .	574
C.3 Involutivity of the Legendre Transform of a Function	574
C.4 Concluding Remarks and Comments	575
Appendix D The Euler–MacLaurin Formula	577
D.1 Bernoulli Numbers	577
D.2 Bernoulli Polynomials	577
D.3 Some Known Operators and Series of Operators	578
D.4 Euler–MacLaurin Series and Formula	578
D.5 The General Euler–MacLaurin Formula	579
D.6 Applications	579
D.7 Again to the Actual Euler–MacLaurin Formula	580
Appendix E Riemann–Stieltjes Integral, Delta Function, and the Concept of Generalized Functions	583
E.1 The Riemann–Stieltjes Integral	583
E.2 Case in Which the Riemann–Stieltjes Integral Reduces to the Riemann Integral	585
E.3 Heaviside Function and an Example of a Riemann–Stieltjes Integral Computation	586
E.4 Generalized Functions	586
E.4.1 Dirac's Delta Function. A Heuristic Description	586
E.5 The Correspondence Between Functions and Functionals	587
E.6 Functionals as Generalized Functions	588
E.7 Differentiation of Generalized Functions	589
E.8 Derivatives of the Heaviside Function and the Delta Function . .	589
Appendix F The Implicit Function Theorem (An Alternative Presentation)	591
F.1 Formulation of the Problem	591
F.2 Some Reminders of Numerical Methods to Solve Equations . . .	591
F.2.1 The Principle of the Fixed Point	593
F.3 The Implicit Function Theorem	595
F.3.1 Statement of the Theorem	595
F.3.2 Proof of the Existence of an Implicit Function	596
F.3.3 Continuity of an Implicit Function	597
F.3.4 Differentiability of an Implicit Function	597
F.3.5 Continuous Differentiability of an Implicit Function . . .	598
F.3.6 Higher Derivatives of an Implicit Function	599

References	601
1 Classic Works	601
1.1 Primary Sources	601
1.2 Major Comprehensive Expository Works	601
1.3 Classical Courses of Analysis from the First Half of the Twentieth Century	602
2 Textbooks	602
3 Classroom Materials	602
4 Further Reading	603
Subject Index	605
Name Index	615