

# Contents

## Part I THE ROOTS OF COMPUTABILITY THEORY

<b>1</b>	<b>Introduction</b>	3
1.1	Algorithms and Computation	3
1.1.1	The Intuitive Concept of the Algorithm and Computation	3
1.1.2	Algorithms and Computations Before the Twentieth Century	5
1.2	Chapter Summary	7
<b>2</b>	<b>The Foundational Crisis of Mathematics</b>	9
2.1	Crisis in Set Theory	9
2.1.1	Axiomatic Systems	9
2.1.2	Cantor's Naive Set Theory	13
2.1.3	Logical Paradoxes	17
2.2	Schools of Recovery	19
2.2.1	Slowdown and Revision	20
2.2.2	Intuitionism	20
2.2.3	Logicism	23
2.2.4	Formalism	26
2.3	Chapter Summary	29
<b>3</b>	<b>Formalism</b>	31
3.1	Formal Axiomatic Systems and Theories	31
3.1.1	What Is a Formal Axiomatic System?	31
3.1.2	Interpretations and Models	35
3.2	Formalization of Logic, Arithmetic, and Set Theory	39
3.3	Chapter Summary	48
<b>4</b>	<b>Hilbert's Attempt at Recovery</b>	49
4.1	Hilbert's Program	49
4.1.1	Fundamental Problems of the Foundations of Mathematics	49
4.1.2	Hilbert's Program	53

- 4.2 The Fate of Hilbert’s Program . . . . . 54
  - 4.2.1 Formalization of Mathematics: Formal Axiomatic System **M** 54
  - 4.2.2 Decidability of **M**: Entscheidungsproblem . . . . . 55
  - 4.2.3 Completeness of **M**: Gödel’s First Incompleteness Theorem 57
  - 4.2.4 Consequences of the First Incompleteness Theorem . . . . . 58
  - 4.2.5 Consistency of **M**: Gödel’s Second Incompleteness Theorem 60
  - 4.2.6 Consequences of the Second Incompleteness Theorem . . . . 61
- 4.3 Legacy of Hilbert’s Program . . . . . 63
- 4.4 Chapter Summary . . . . . 64
- Problems . . . . . 64
- Bibliographic Notes . . . . . 65

**Part II CLASSICAL COMPUTABILITY THEORY**

- 5 The Quest for a Formalization . . . . . 69**
  - 5.1 What Is an Algorithm and What Do We Mean by Computation? . . . 69
    - 5.1.1 Intuition and Dilemmas . . . . . 70
    - 5.1.2 The Need for Formalization . . . . . 71
  - 5.2 Models of Computation . . . . . 72
    - 5.2.1 Modelling After Functions . . . . . 72
    - 5.2.2 Modelling After Humans . . . . . 80
    - 5.2.3 Modelling After Languages . . . . . 82
    - 5.2.4 Reasonable Models of Computation . . . . . 86
  - 5.3 Computability (Church-Turing) Thesis . . . . . 87
    - 5.3.1 History of the Thesis . . . . . 87
    - 5.3.2 The Thesis . . . . . 88
    - 5.3.3 Difficulties with Total Functions . . . . . 90
    - 5.3.4 Generalization to Partial Functions . . . . . 93
    - 5.3.5 Applications of the Thesis . . . . . 97
  - 5.4 Chapter Summary . . . . . 97
  - Problems . . . . . 98
  - Bibliographic Notes . . . . . 99
- 6 The Turing Machine . . . . . 101**
  - 6.1 Turing Machine . . . . . 101
    - 6.1.1 Basic Model . . . . . 102
    - 6.1.2 Generalized Models . . . . . 107
    - 6.1.3 Equivalence of Generalized and Basic Models . . . . . 109
    - 6.1.4 Reduced Model . . . . . 113
    - 6.1.5 Equivalence of Reduced and Basic Models . . . . . 114
    - 6.1.6 Use of Different Models . . . . . 114
  - 6.2 Universal Turing Machine . . . . . 115
    - 6.2.1 Coding and Enumeration of Turing Machines . . . . . 115
    - 6.2.2 The Existence of a Universal Turing Machine . . . . . 117
    - 6.2.3 The Importance of the Universal Turing Machine . . . . . 119

6.2.4	Practical Consequences: Data vs. Instructions . . . . .	119
6.2.5	Practical Consequences: General-Purpose Computer . . . . .	119
6.2.6	Practical Consequences: Operating System . . . . .	121
6.2.7	Practical Consequences: RAM Model of Computation . . . . .	122
6.3	Use of a Turing Machine . . . . .	125
6.3.1	Function Computation . . . . .	125
6.3.2	Set Generation . . . . .	127
6.3.3	Set Recognition . . . . .	130
6.3.4	Generation vs. Recognition . . . . .	133
6.3.5	The Standard Universes $\Sigma^*$ and $\mathbb{N}$ . . . . .	136
6.3.6	Formal Languages vs. Sets of Natural Numbers . . . . .	137
6.4	Chapter Summary . . . . .	138
	Problems . . . . .	139
	Bibliographic Notes . . . . .	141
<b>7</b>	<b>The First Basic Results</b> . . . . .	<b>143</b>
7.1	Some Basic Properties of Semi-decidable (C.E.) Sets . . . . .	143
7.2	Padding Lemma and Index Sets . . . . .	145
7.3	Parametrization (s-m-n) Theorem . . . . .	147
7.3.1	Deduction of the Theorem . . . . .	148
7.4	Recursion (Fixed-Point) Theorem . . . . .	149
7.4.1	Deduction of the Theorem . . . . .	150
7.4.2	Interpretation of the Theorem . . . . .	151
7.4.3	Fixed Points of Functions . . . . .	152
7.4.4	Practical Consequences: Recursive Program Definition . . . . .	153
7.4.5	Practical Consequences: Recursive Program Execution . . . . .	154
7.4.6	Practical Consequences: Procedure Calls in General-Purpose Computers . . . . .	156
7.5	Chapter Summary . . . . .	158
	Problems . . . . .	159
	Bibliographic Notes . . . . .	160
<b>8</b>	<b>Incomputable Problems</b> . . . . .	<b>161</b>
8.1	Problem Solving . . . . .	161
8.1.1	Decision Problems and Other Kinds of Problems . . . . .	162
8.1.2	Language of a Decision Problem . . . . .	163
8.1.3	Subproblems of a Decision Problem . . . . .	165
8.2	There Is an Incomputable Problem — Halting Problem . . . . .	166
8.2.1	Consequences: The Basic Kinds of Decision Problems . . . . .	169
8.2.2	Consequences: Complementary Sets and Decision Problems . . . . .	171
8.2.3	Consequences: There Is an Incomputable Function . . . . .	172
8.3	Some Other Incomputable Problems . . . . .	172
8.3.1	Problems About Turing Machines . . . . .	173
8.3.2	Post's Correspondence Problem . . . . .	175
8.3.3	Problems About Algorithms and Computer Programs . . . . .	175

8.3.4	Problems About Programming Languages and Grammars . .	177
8.3.5	Problems About Computable Functions . . . . .	179
8.3.6	Problems from Number Theory . . . . .	180
8.3.7	Problems from Algebra . . . . .	180
8.3.8	Problems from Analysis . . . . .	182
8.3.9	Problems from Topology . . . . .	183
8.3.10	Problems from Mathematical Logic . . . . .	184
8.3.11	Problems About Games . . . . .	185
8.4	Can We Outwit Incomputable Problems? . . . . .	187
8.5	Chapter Summary . . . . .	189
	Problems . . . . .	189
	Bibliographic Notes . . . . .	190
<b>9</b>	<b>Methods of Proving Incomputability . . . . .</b>	<b>191</b>
9.1	Proving by Diagonalization . . . . .	191
9.1.1	Direct Diagonalization . . . . .	191
9.1.2	Indirect Diagonalization . . . . .	194
9.2	Proving by Reduction . . . . .	196
9.2.1	Reductions in General . . . . .	196
9.2.2	The $m$ -Reduction . . . . .	197
9.2.3	Undecidability and $m$ -Reduction . . . . .	199
9.2.4	The 1-Reduction . . . . .	201
9.3	Proving by the Recursion Theorem . . . . .	204
9.4	Proving by Rice's Theorem . . . . .	205
9.4.1	Rice's Theorem for Functions . . . . .	205
9.4.2	Rice's Theorem for Index Sets . . . . .	206
9.4.3	Rice's Theorem for Sets . . . . .	208
9.4.4	Consequences: Behavior of Abstract Computing Machines . .	209
9.5	Incomputability of Other Kinds of Problems . . . . .	210
9.6	Chapter Summary . . . . .	213
	Problems . . . . .	213
	Bibliographic Notes . . . . .	215
 <b>Part III RELATIVE COMPUTABILITY</b>		
<b>10</b>	<b>Computation with External Help . . . . .</b>	<b>219</b>
10.1	Turing Machines with Oracles . . . . .	219
10.1.1	Turing's Idea of Oracular Help . . . . .	220
10.1.2	The Oracle Turing Machine ( $o$ -TM) . . . . .	223
10.1.3	Some Basic Properties of $o$ -TMs . . . . .	225
10.1.4	Coding and Enumeration of $o$ -TMs . . . . .	226
10.2	Computation with Oracles . . . . .	228
10.2.1	Generalization of Classical Definitions . . . . .	228
10.2.2	Convention: The Universe $\mathbb{N}$ and Single-Argument Functions	231
10.3	Other Ways to Make External Help Available . . . . .	231

10.4	Relative Computability Thesis	232
10.5	Practical Consequences: $\sigma$ -TM with a Database or Network	232
10.6	Practical Consequences: Online and Offline Computation	233
10.7	Chapter Summary	234
	Bibliographic Notes	234
<b>11</b>	<b>Degrees of Unsolvability</b>	<b>235</b>
11.1	Turing Reduction	235
11.1.1	Turing Reduction of a Computational Problem	236
11.1.2	Some Basic Properties of the Turing Reduction	237
11.2	Turing Degrees	240
11.3	Chapter Summary	243
	Problems	244
	Bibliographic Notes	244
<b>12</b>	<b>The Turing Hierarchy of Unsolvability</b>	<b>245</b>
12.1	The Perplexities of Unsolvability	245
12.2	The Turing Jump	246
12.2.1	Properties of the Turing Jump of a Set	247
12.3	Hierarchies of $T$ -Degrees	249
12.3.1	The Jump Hierarchy	250
12.4	Chapter Summary	252
	Problems	252
	Bibliographic Notes	253
<b>13</b>	<b>The Class <math>\mathcal{D}</math> of Degrees of Unsolvability</b>	<b>255</b>
13.1	The Structure $(\mathcal{D}, \leq, ')$	255
13.2	Some Basic Properties of $(\mathcal{D}, \leq, ')$	257
13.2.1	Cardinality of Degrees and of the Class $\mathcal{D}$	257
13.2.2	The Class $\mathcal{D}$ as a Mathematical Structure	258
13.2.3	Intermediate $T$ -Degrees	263
13.2.4	Cones	264
13.2.5	Minimal $T$ -Degrees	266
13.3	Chapter Summary	267
	Problems	267
	Bibliographic Notes	267
<b>14</b>	<b>C.E. Degrees and the Priority Method</b>	<b>269</b>
14.1	C.E. Turing Degrees	269
14.2	Post's Problem	270
14.2.1	Post's Attempt at a Solution to Post's Problem	271
14.3	The Priority Method and Priority Arguments	274
14.3.1	The Priority Method in General	274
14.3.2	The Friedberg-Muchnik Solution to Post's Problem	278
14.3.3	Priority Arguments	279
14.4	Some Properties of C.E. Degrees	279

14.5 Chapter Summary .....	280
Problems .....	280
Bibliographic Notes .....	281
<b>15 The Arithmetical Hierarchy .....</b>	<b>283</b>
15.1 Decidability of Relations .....	283
15.2 The Arithmetical Hierarchy .....	284
15.3 The Link with the Jump Hierarchy .....	288
15.4 Practical Consequences: Proving the Incomputability .....	290
15.5 Chapter Summary .....	292
Problems .....	292
Bibliographic Notes .....	293
<b>16 Further Reading .....</b>	<b>295</b>
<b>A Mathematical Background .....</b>	<b>297</b>
<b>B Notation Index .....</b>	<b>305</b>
<b>References .....</b>	<b>311</b>
<b>Index .....</b>	<b>319</b>